Rotation, moment of inertia

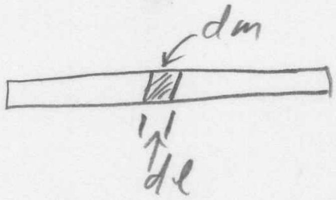
A thin rod of length L has a non-uniform density profile of $\overset{\lambda}{\cancel{\alpha}} = \overset{\lambda_0}{\cancel{\alpha_0}}\left[2\frac{l^2}{L^2}+\frac{1}{3}\right]$.

a) Calculate the total mass of the rod.

b) Calculate the center of mass of the rod.

c) The rod is stood on it's end (heavy end to the top) so that it is perpendicular to the floor. It is allowed to fall without slipping. What is it's angular velocity as it hits the floor?

a) $0^{\underline{th}}$ moment, total mass



$$M = \int_0^L dm$$

$$dm = \lambda\, dl$$

$$dm = \lambda_0\left[2\frac{l^2}{L^2}+\frac{1}{3}\right]dl$$

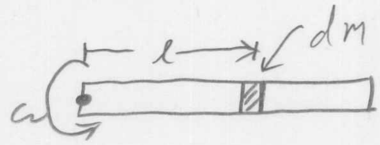
$$M = \lambda_0\int_0^L \left[2\frac{l^2}{L^2}+\frac{1}{3}\right]dl$$

$$= \lambda_0\left(\frac{2}{\cancel{L^2}}\frac{1}{3}L^{\cancel{3}} + \frac{1}{3}L\right) = \lambda_0 L\left(\frac{2}{3}+\frac{1}{3}\right)$$

$$\boxed{M = \lambda_0 L}$$

c) 2^{nd} moment – Moment of Inertia

$$I = \int l^2 dm, \quad dm = \lambda_0\left[2\frac{l^2}{L^2} + \frac{1}{3}\right]dl$$

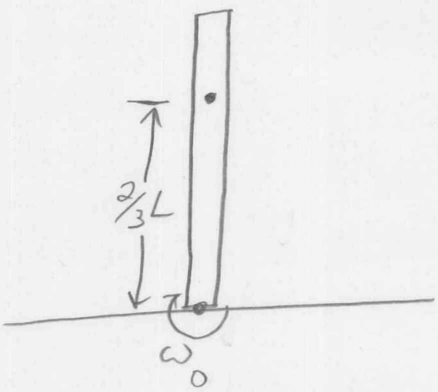


$$I = \int_0^L l^2 \lambda_0\left[2\frac{l^2}{L^2} + \frac{1}{3}\right]dl$$

$$= \lambda_0 \int \left[2\frac{l^4}{L^2} + \frac{l^2}{3}\right]dl$$

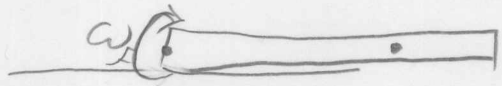
$$= \lambda_0\left(\frac{2}{\cancel{L^2}}\frac{1}{5}L^{\cancel{5}\,3} + \frac{1}{3}L^3\right)$$

$$\boxed{I = \frac{11}{15}\lambda_0 L^3}$$



$U_I = mg\,\frac{2}{3}L, \quad U_F = 0$

$K_I = 0 \qquad K_F = \frac{1}{2}I\omega^2$



From pt a, $M = \lambda_0 L$ so:

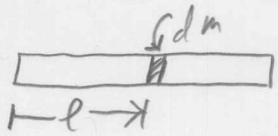
$$\cancel{\lambda_0}\cancel{L}\,g\,\frac{2}{3}\cancel{L} = \frac{1}{\cancel{2}}\,\frac{11}{15}\cancel{\lambda_0}L^{\cancel{3}}\omega^2$$

$$\boxed{\omega = \left(\frac{15}{33}\frac{g}{L}\right)^{1/2}}$$

b) 1^{st} moment divided By 0^{th} moment = Center of mass

$$x_{cm} = \frac{1}{M}\int x\, dm,$$ From part a,

$$dm = \lambda_0\left[2\frac{\ell^2}{L^2} + \frac{1}{3}\right]d\ell$$



$$\ell_{cm} = \frac{1}{M}\int \ell\,\lambda_0\left[2\frac{\ell^2}{L^2} + \tfrac{1}{3}\right]d\ell,$$ From part a, $M = \lambda_0 L$

$$= \frac{\cancel{\lambda_0}}{\cancel{\lambda_0}L}\int_0^L \left[2\frac{\ell^3}{L^2} + \frac{\ell}{3}\right]d\ell$$

$$= \frac{1}{L}\left(\frac{\cancel{2}1}{\cancel{4}2}\frac{L^{\cancel{4}2}}{\cancel{L^2}} + \frac{1}{2}\frac{L^2}{3}\right)$$

$$= \frac{L^{\cancel{2}}}{\cancel{L}}\left(\tfrac{1}{2} + \tfrac{1}{6}\right) = \tfrac{2}{3}L$$

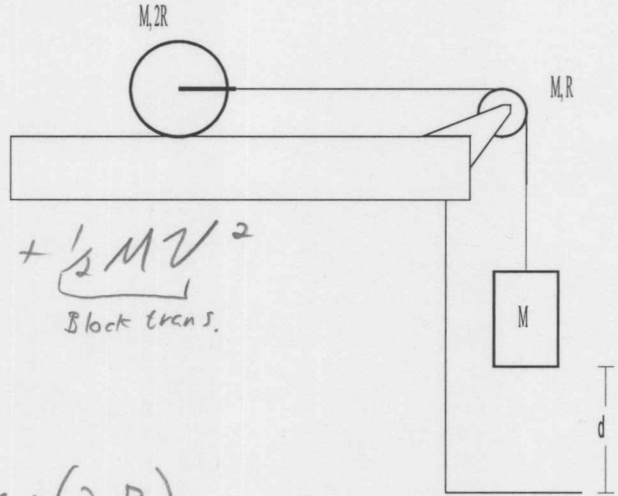
$$\boxed{\ell_{cm} = \tfrac{2}{3}L}$$

Rotation, moment of inertia

A solid cylinder (radius = R, mass = M) rolls without slipping as it is pulled by a massless yoke attached to a string. The string goes over a frictionless pulley shaped as a solid disk (radius = R, mass = M) and is attached to a mass (m = M). What is the velocity of the system after the hanging mass has fallen a distance d?

USE ENERGY to solve this problem.

a) What is the velocity of the hanging mass if the pulley is massless and frictionless
b) What is the velocity of the hanging mass if the pully is a flat disk with mass m and radius ½R.



a)

$U_I = Mgd, \quad U_F = 0$

$K_I = 0 \quad K_F = \underbrace{\tfrac{1}{2}I\omega^2}_{\text{disk rot}} + \underbrace{\tfrac{1}{2}Mv^2}_{\text{disk trans.}} + \underbrace{\tfrac{1}{2}Mv^2}_{\text{Block trans.}}$

$I_{disk} = \tfrac{1}{2}M(2R)^2$

$I = 2MR^2, \quad v = \omega(2R)$

$\Rightarrow \omega = \dfrac{v}{2R}$

So:

$\cancel{M}gd = \tfrac{1}{\cancel{2}}\cancel{2}\cancel{M}\cancel{R^2}\dfrac{v^2}{4\cancel{R^2}} + \tfrac{1}{2}\cancel{M}v^2 + \tfrac{1}{2}\cancel{M}v^2$

$gd = \tfrac{1}{4}v^2 + \tfrac{1}{2}v^2 + \tfrac{1}{2}v^2 = \tfrac{5}{4}v^2$

$\boxed{v = (\tfrac{4}{5}gd)^{\frac{1}{2}}}$

(b)

Same thing, but add second pulley to K_F

$$K_F = \underbrace{\tfrac{1}{2} I_1 \omega_1^2}_{\text{disk 1 rot}} + \underbrace{\tfrac{1}{2} M v^2}_{\text{disk 1 trans}} + \underbrace{\tfrac{1}{2} I_2 \omega_2^2}_{\text{disk 2 rot}} + \underbrace{\tfrac{1}{2} M v^2}_{\text{Block trans}}$$

$$I_2 = \tfrac{1}{2} M R^2, \quad \omega_2 = \frac{v}{R}$$

$$\cancel{M} g d = \tfrac{1}{\cancel{2}} \cancel{2} \cancel{M} \cancel{R^2} \frac{v^2}{4 \cancel{R^2}} + \tfrac{1}{2} \cancel{M} v^2 + \tfrac{1}{2} \tfrac{1}{2} \cancel{M} \cancel{R^2} \frac{v^2}{\cancel{R^2}} + \tfrac{1}{2} \cancel{M} v^2$$

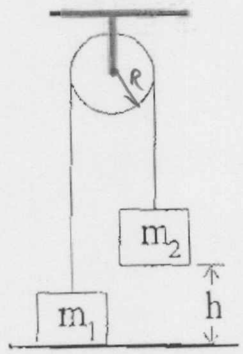
$$g d = \tfrac{1}{4} v^2 + \tfrac{1}{2} v^2 + \tfrac{1}{4} v^2 + \tfrac{1}{2} v^2$$

$$g d = \tfrac{3}{2} v^2 \Rightarrow \boxed{v^2 = \left(\tfrac{2}{3} g d\right)^{1/2}}$$

Rotation, moment of inertia

Use work energy to solve the following problem.
Two masses are connected by a light string passing over a frictionless pulley. the Mass m_2 is released from rest at a height of 4.0 m above the ground. You can treat the pulley as a solid disk.

Determine the speed of m_1 as m_2 hits the ground.



$m_1 = 3.0$ kg
$m_2 = 5.0$ kg
$m_{pulley} = 0.5$ kg
$r_{pulley} = 0.1$ m

$U_I = m_2 g h \qquad U_F = m_1 g h$

$K_I = 0 \qquad K_F = \frac{1}{2} m_1 v^2 + \frac{1}{2} m_2 v^2 + \frac{1}{2} I \omega^2$

$$m_2 g h = m_1 g h + \frac{1}{2} m_1 v^2 + \frac{1}{2} m_2 v^2 + \frac{1}{2}\left(\frac{1}{2} M_P \cancel{R^2}\right) \frac{v^2}{\cancel{R^2}}$$

$$(m_2 - m_1) g h = \left(\frac{1}{2} m_1 + \frac{1}{2} m_2 + \frac{1}{4} m_P\right) v^2$$

$$\boxed{v^2 = \left(\frac{m_2 - m_1}{\frac{1}{2} m_1 + \frac{1}{2} m_2 + \frac{1}{4} m_P}\right) g h}$$