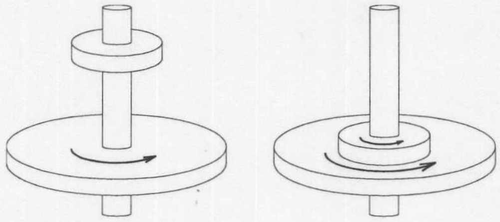
Rotation, moment of inertia

A disk with moment of inertia of I_1 rotates about a vertical, frictionless axle with an angular speed ω_o. A second disk, initially at rest, has a moment of inertia I_2 and is dropped onto the first disk. Because of friction between the two surfaces, the two disks eventually reach the same speed ω_f.

(a) Calculate ω_f.

(b) Show that the kinetic energy of the system decreases in this interaction and calculate the ratio of the final rotational energy to the initial rotational energy.



a) $L_I = L_F$ conserve angular momentum

$$I_1 \omega_0 = (I_1 + I_2)\,\omega_f$$

$$\boxed{\omega_f = \frac{I_1}{I_1 + I_2}\,\omega_0}$$

b) $K_I = \frac{1}{2} I_1 \omega_0^2$, $\quad K_F = \frac{1}{2}(I_1 + I_2)\,\omega_f^2$

$$= \frac{1}{2}\cancel{(I_1 + I_2)}\,\frac{I_1^2}{(I_1+I_2)^{\cancel{2}}}\,\omega_0^2$$

$$\frac{K_I}{K_F} = \frac{\cancel{\frac{1}{2}}\,\cancel{I_1}\,\cancel{\omega_0^2}}{\cancel{\frac{1}{2}}\,I_1^{\cancel{2}}/(I_1+I_2)\,\cancel{\omega_0^2}} = \frac{I_1 + I_2}{I_1} > 1$$

So $\boxed{K_I > K_F}$

Rotation, moment of inertia

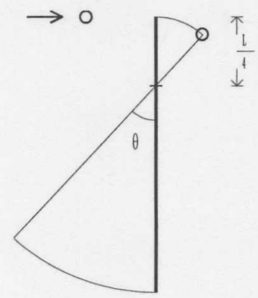
A chunk of clay of mass $m_{clay} = m = 0.500$ kg moving at a velocity of 3.00 m/s hits and sticks to a long thin rod that spins freely about a point ¼ of the way down from the top. The rod has a mass of $m_{rod} = 2m = 1.00$ kg and a length of 1.00 m.

Note: The moment of inertia a long thin rod about its *center of mass* is $I_{cm} = \frac{1}{12} mL^2$.

(a) What is the moment of inertia of the rod–clay combination?

(b) What is ω right after the collision?

(c) What is θ_{max}?



a) $I_T = I_{Rod} + I_{clay}$

$$I_{Rod} = I_{cmRod} + m_{Rod} d^2$$

$$= \frac{1}{12} m_{Rod} L^2 + m_{Rod}\left(\frac{L^2}{4^2}\right)$$

$$= \left(\frac{1}{12} + \frac{1}{16}\right) m_{Rod} L^2$$

$$\boxed{I_{Rod} = \frac{7}{48} m_{Rod} L^2}$$

$$I_{clay} = m_{clay}\left(\frac{L}{4}\right)^2$$

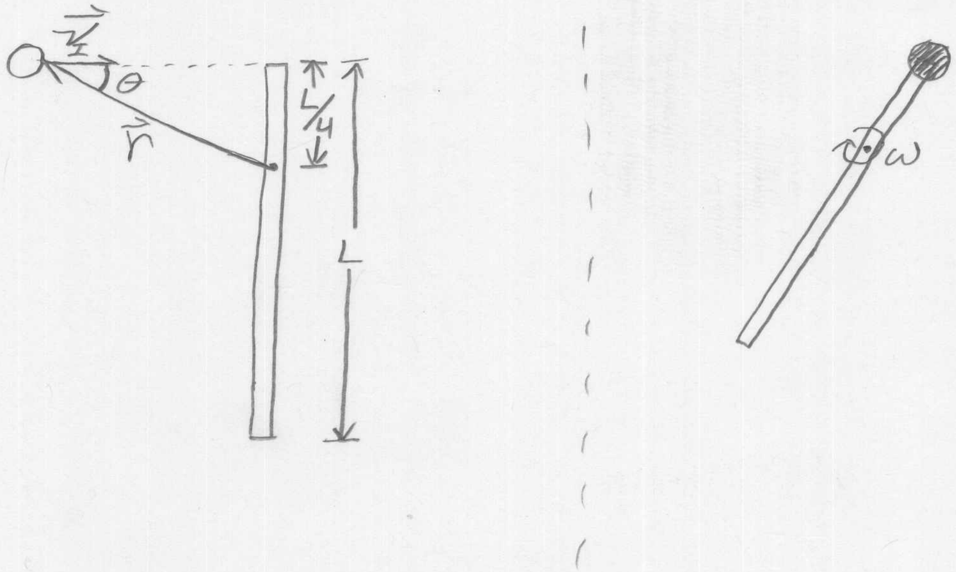
$$\boxed{I_{clay} = \frac{1}{16} m_{clay} L^2}$$

Since $m_{clay} = m$ and $m_{Rod} = 2m$

$$I_T = \frac{7}{\cancel{48}_{24}} 2mL^2 + \frac{1}{16} mL^2.$$

$$\boxed{I_T = \frac{17}{48} mL^2}$$

(b) Conserve Angular Momentum



$$L_I = L_F$$

$$m(\vec{r} \times \vec{v}_I) = I_T \omega$$

$$m r v_I \sin\theta = I_T \omega$$

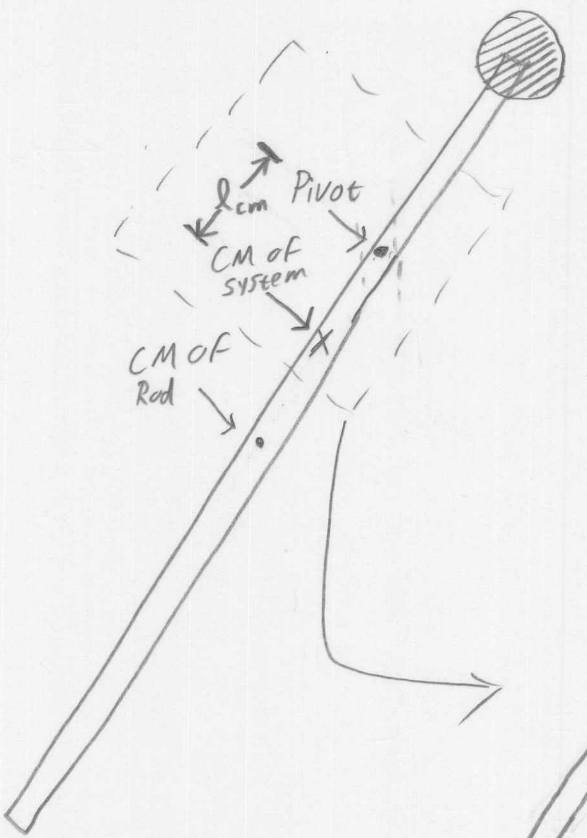
$$m v_I \frac{L}{4} = I_T \omega$$

$$\Rightarrow \omega = \frac{m v_I L}{4 I_T}$$

$$\omega = \frac{\cancel{48}^{\,12}}{17} \frac{\cancel{m} v_I \cancel{L}}{\cancel{4} \cancel{m} L^{\cancel{2}}}$$

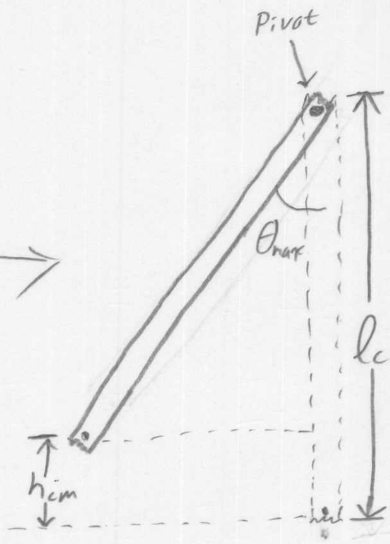
$$\boxed{\omega = \frac{12}{17} \frac{v_I}{L}}$$

c)



Remember that gravitational Potential for an extended object goes as:

$$\underline{mgh_{cm} = U_g}$$



$$\underline{h_{cm} = l_{cm} - l_{cm}\cos\theta}$$

$U_I = 0$ $\qquad K_I = \frac{1}{2} I_T \omega^2$

$U_F = mgh_{cm}$ $\qquad K_F = 0$

$$m_T g h_{cm} = \frac{1}{2} I_T \omega^2$$

$$l_{cm} = \frac{1}{m_T}\left(-m_{clay}\frac{L}{4} + m_{Rod}\frac{L}{4}\right)$$

$$= \frac{1}{m_T}\left(-\frac{1}{4}mL + \frac{1}{\cancel{4}_2}2mL\right), \qquad m_T = m_{Rod} + m_{clay}$$

$$= \frac{1}{3\cancel{m}}\left(\frac{-1}{4} + \frac{1}{2}\right)\cancel{m}L \qquad \underline{m_T = 3m}$$

$$\boxed{l_{cm} = \frac{1}{12}L}$$

continued ↓

Put it all together

$$m_T g h_{cm} = \tfrac{1}{2} I_T \omega^2 , \quad m_T = 3m , \quad h_{cm} = l_{cm} - l_{cm}\cos\theta_m$$

$$l_{cm} = \tfrac{1}{12} L , \quad I_T = \tfrac{17}{48} mL^2 , \quad \omega = \tfrac{12}{17}\frac{v_I}{L}$$

$$\downarrow$$

$$3\cancel{m} g\, l_{cm}(1-\cos\theta_m) = \frac{1}{2}\frac{\cancel{17}}{48}\cancel{m}\cancel{L^2}\left(\frac{12^2}{17^2}\right)\frac{v_I^2}{\cancel{L^2}}$$

$$3g\frac{1}{\cancel{12}_{4}}L(1-\cos\theta_m) = \frac{1}{\cancel{2}}\frac{12^2}{\cancel{2}\cdot\cancel{12}} v_I^2$$

$$gL(1-\cos\theta_m) = 12 v_I^2$$

$$\boxed{\cos\theta_m = 1 - \frac{12 v_I^2}{gL}}$$

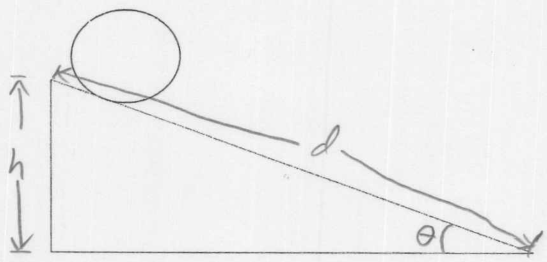
Rotation, moment of inertia

An object with a radius R and moment of inertia $I = cMR^2$ rolls without slipping a distance d down an incline plane that makes an angle θ with the horizontal. What is it's linear velocity at the bottom?

a) Use energy to solve this problem

b) Use torque and Kinematics to solve this problem and verify the answers are the same.

c) What is the minimum value of μ_s required for the wheel to not slip.



a) $U_i = mgh = mgd\sin\theta$ $\quad K_i = 0$

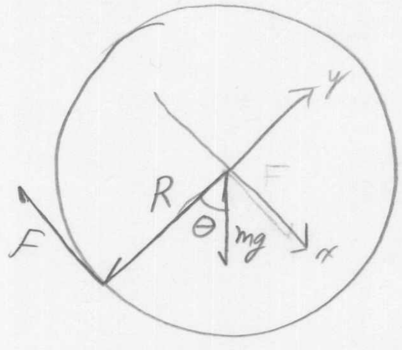
$U_f = 0$ $\quad K_f = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$

$$\cancel{m}gd\sin\theta = \frac{1}{2}\cancel{m}v^2 + \frac{1}{2}c\cancel{m}\cancel{R^2}\frac{v^2}{\cancel{R^2}}$$

$$gd\sin\theta = \frac{1}{2}v^2(1+c)$$

$$\boxed{v = \left[\frac{2gd\sin\theta}{(1+c)}\right]^{1/2}}$$

(b)



① $T = fR = I\alpha$

② $F - f = mg\sin\theta - f = ma$

since we want translational velocity, we'll change α into a: $a = R\alpha \Rightarrow \alpha = \dfrac{a}{R}$

$$f\cancel{R} = cmR^{\cancel{2}}\frac{a}{\cancel{R}} \Rightarrow f = cma$$

Plug into ②

$$\cancel{m}g\sin\theta - c\cancel{m}a = ma$$

$$g\sin\theta = a(1+c)$$

$$\boxed{a = \frac{g\sin\theta}{(1+c)}}$$

Kinematics

$$d = \cancel{d_0}^{0} + \cancel{v_0 t}^{0} + \tfrac{1}{2}at^2$$

$$d = \frac{1}{2}\frac{g\sin\theta}{1+c}t^2 \Rightarrow t^2 = \frac{2d(1+c)}{g\sin\theta}$$

$$v = \cancel{v_0}^{0} + at$$

$$v = \frac{g\sin\theta}{(1+c)}\cdot\left[\frac{2d(1+c)}{g\sin\theta}\right]^{1/2}$$

$$\boxed{v = \left[\frac{2dg\sin\theta}{(1+c)}\right]^{1/2}}$$ same as part (a)

$N = mg\cos\theta$, $F = cma = cm$ [illegible] $(1+c)$

so when $c\cancel{m}a \geq \cancel{m}g\cos\theta\,\mu_s$

$$\frac{c}{(1+c)} \cancel{g}\sin\theta \geq \cancel{g}\cos\theta\,\mu_s$$

$$\boxed{\mu_s \leq \frac{c}{(1+c)}\tan\theta}$$