Proof of The Work Energy Theorem

Start with the definition of work done by a force

$$W_i = \int \vec{F}_i \cdot d\vec{s} \quad , \qquad \vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}, \; d\vec{s} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

$$W_i = \int \left(F_{xi}\hat{i} + F_{yi}\hat{j} + F_{zi}\hat{k}\right)\left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right)$$

$$= \int \left(F_{xi}dx + F_{yi}dy + F_{zi}dz\right)$$

$$W_i = \int F_{xi}dx + \int F_{yi}dy + \int F_{zi}dz$$

Now, the <u>net</u> work is the sum of the individual works

$$W_{net} = \sum_{i=1}^{n} W_i = \sum_{i=1}^{n}\left(\int F_{xi}dx + \int F_{yi}dy + \int F_{zi}dz\right)$$

$$= \sum_{i=1}^{n}\int F_{xi}dx + \sum_{i=1}^{n}\int F_{yi}dy + \sum_{i=1}^{n}\int F_{zi}dz$$

$$= \int \sum_{i=1}^{n} F_{xi}\,dx + \int \sum_{i=1}^{n} F_{yi}dy + \int \sum_{i=1}^{n} F_{zi}dz$$

$$W_{net} = \int F_{xnet}dx + \int F_{ynet}dy + \int F_{znet}dz$$

$$\int_{x_0}^{x_1} F_{xnet}dx = \int_{x_0}^{x_1} ma_x dx = \int_{x_0}^{x_1} m\frac{dv_x}{dt}dx = \int_{v_0}^{v_1} m\frac{dx}{dt}dv_x = \int_{v_0}^{v_1} m v_x\, dv_x$$

$$= \tfrac{1}{2}mv_{1x}^2 - \tfrac{1}{2}mv_{0x}^2 = \Delta K_x \Longrightarrow \boxed{W_{net} = \Delta K} \, *$$