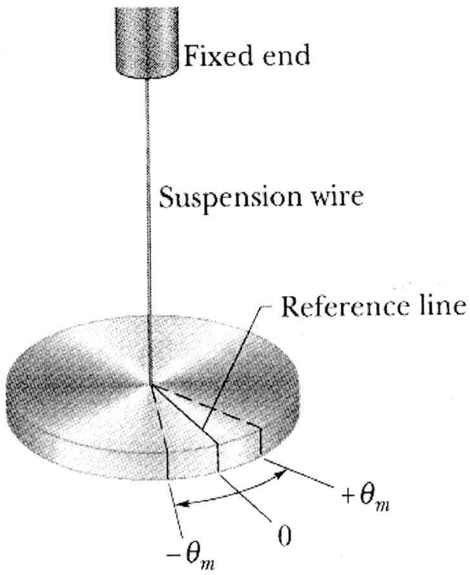
Oscillation – Set 2

The device in the picture below is known as a torsion pendulum. It is a flat disk attached to a length of stiff wire. When the wire is twisted, it responds by providing a torque on the disk, much the same way a spring provides a force when it is stretched. The torque provided by the wire is $T = -\kappa\theta$, where κ (greek letter kappa) is the torsion constant and theta is the angular displacement from equilibrium.

a) The moment of Inertia of the disk is $I = \frac{1}{2}MR^2$. Using the rotational version of Newton's Second Law, find the oscillator frequency of the torsion pendulum.

b) If a solid bar of length L, $I = \frac{1}{12}ML^2$, were suspended from the wire, what would the oscillator frequency be?



Let's solve the problem generally in terms of I, then plug in the different I for each object.

NSL for rotation

$$\sum T = I\alpha$$

$$-k\theta = I\frac{d^2\theta}{dt^2}$$

$$\Rightarrow \boxed{\frac{d^2\theta}{dt^2} = -\frac{k}{I}\theta}$$ SHO!

This is the simple harmonic oscillator equation, except in θ instead of x.

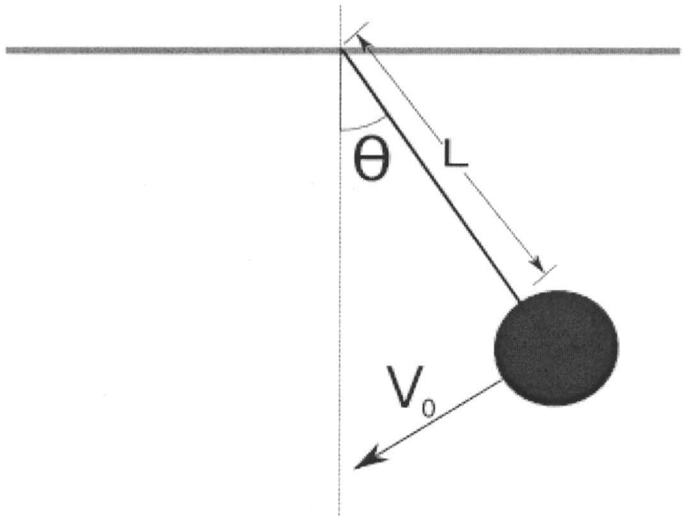
$$\boxed{\omega = \left(\frac{k}{I}\right)^{1/2}}$$

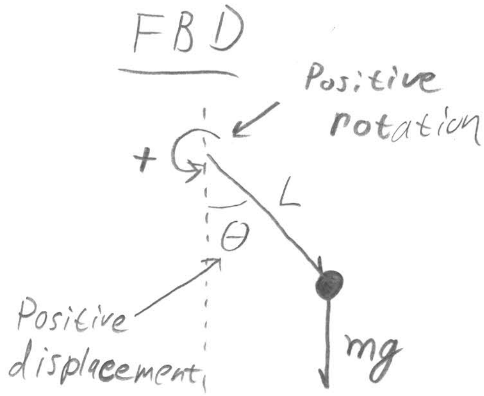
a) $\omega = \left[\frac{2k}{MR^2}\right]^{1/2}$ b) $\omega = \left[\frac{12k}{ML^2}\right]^{1/2}$

Oscillation – Set 2   2

Below is a simple pendulum consisting of a massless rod of length L with a point mass if mass m attached to the end.

a) Find the frequency of small oscillations of the pendulum.
b) At t=0, the pendulum makes an angle θ_0 with the vertical and the point mass has a velocity V_0. What is ~~the amplitude of the oscillator?~~ Phase angle?





$\vec{L} \times m\vec{g} \Rightarrow \otimes$
gives negative torque by rhr

NSL

$\sum \tau = I\alpha, \quad I = mL^2$

$-\cancel{m}g\cancel{L}\sin\theta = \cancel{m}L^{\cancel{2}}\dfrac{d^2\theta}{dt^2}$

$\Rightarrow \dfrac{d^2\theta}{dt^2} = -\dfrac{g}{L}\sin\theta$, Almost a SHO.

For small θ, $\sin\theta \cong \theta$ (small angle approximation)

So; For small oscillations : $\boxed{\dfrac{d^2\theta}{dt^2} = -\dfrac{g}{L}\theta}$ SHO!

$\boxed{\omega = \sqrt{\dfrac{g}{L}}}$

continued ↓

Oscillation Set 2, P2 continued

b) $\theta(0) = \theta_0$, $v(0) = V_0 = \omega_0 L$

!!Be very careful!!

This angular velocity is not the same ω as the oscillator frequency, $\sqrt{g/L}$. Let's call $\boxed{\sqrt{g/L} = \omega_F}$

Angular versions of SHO general solution

$$\theta(t) = A\cos(\omega_F t + \phi)$$
$$\omega(t) = -\omega_F A \sin(\omega_F t + \phi)$$

$$\theta_0 = A\cos(\phi)$$

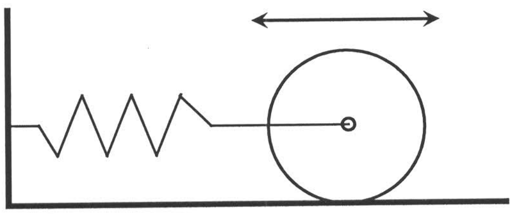
$$\frac{V_0}{L} = -\omega_F A \sin(\phi)$$

$$\Rightarrow \frac{V_0}{L\theta_0} = \frac{-\omega_F \cancel{A} \sin(\phi)}{\cancel{A}\cos(\phi)}$$

$$\Rightarrow \boxed{\tan(\phi) = -\frac{V_0}{\omega_F L \theta_0}}$$

Oscillation – Set 2

A solid cylinder of mass M is attached to a horizontal spring with force constant k. The cylinder can roll without slipping along the horizontal plane. When the system is displaced from the equilibrium position, it executes simple harmonic motion. Derive an expression for the period of the oscillations in terms of M, k, and I.

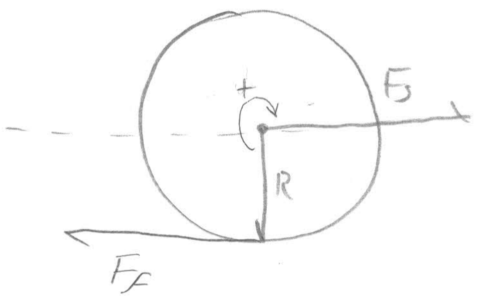


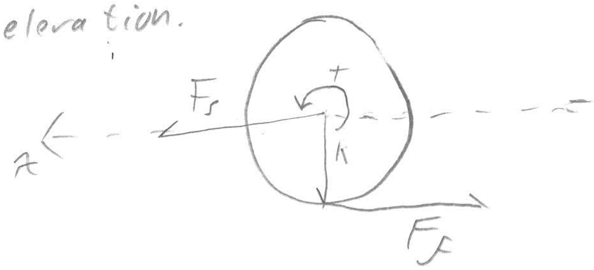
FBD – Depends on which side of equilibrium you're on.

Left of equilibrium

Right of equilibrium

Let + be in the direction of acceleration.





NSL

$$\Sigma T = I\alpha \qquad \Sigma F = ma$$

$$RF_f = I\alpha \qquad F_s - F_f = ma$$

$$\Rightarrow F_f = \frac{I}{R}\alpha \qquad \Rightarrow F_s - \frac{I}{R}\alpha = ma, \quad F_s = -kx$$

$$-kx = \frac{I}{R}\alpha + ma$$

Now, rotation and translation should give the same answer...

continued ↓

Oscillation Set 2, P3 continued

Rotation

$$x = R\theta, \quad a = R\alpha$$

$$-kR\theta = \frac{I}{R}\alpha + mR\alpha \Rightarrow -k\theta = \left[\frac{I}{R^2} + m\right]\alpha$$

and $\alpha = \frac{d^2\theta}{dt^2}$ so:

$$\frac{d^2\theta}{dt^2} = -\frac{k}{I/R^2 + m}\theta \Rightarrow \boxed{\omega = \left[\frac{k}{I/R^2 + m}\right]^{1/2}}$$

Translation

$$\alpha = \frac{a}{R}$$

$$-kx = \frac{I}{R}\cdot\frac{a}{R} + ma \Rightarrow -kx = \left(\frac{I}{R^2} + m\right)a$$

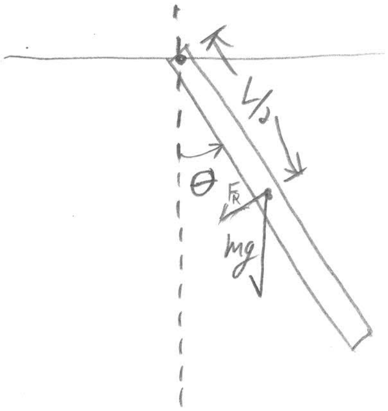
and $a = \frac{d^2x}{dt^2}$ so:

$$\frac{d^2x}{dt^2} = -\frac{k}{I/R^2 + m}x \Rightarrow \boxed{\omega = \left[\frac{k}{I/R^2 + m}\right]^{1/2}}$$

Oscillation – Set 2

A meter stick with a mass M is suspended from one end and allowed to swing like a pendulum.

a) What is its **period** of small oscillations?
b) What length *L* does a simple pendulum (a point mass attached to a massless rod) need in order to have the same period?



$I = \frac{1}{3}ML^2$, $F = -mg\sin\theta$, $L = 1$ meter

$\Sigma T = I\alpha$

$-(mg\sin\theta)\frac{L}{2} = I\alpha$

$-mgL\sin\theta = 2I\alpha$

$-\cancel{m}g\cancel{L}\sin\theta = 2\cdot\frac{1}{3}\cancel{m}L^{\cancel{2}}\frac{d^2\theta}{dt^2}$

$$\boxed{\frac{d^2\theta}{dt^2} = -\frac{3}{2}\frac{g}{L}\sin\theta}$$

a) For small oscillations, $\sin\theta \cong \theta$

$\Rightarrow \frac{d^2\theta}{dt^2} = -\frac{3}{2}\frac{g}{L}\theta \Rightarrow \omega = \left(\frac{3}{2}\frac{g}{L}\right)^{1/2}$

and $T = \frac{2\pi}{\omega}$, $L = 1$ m, $T = 2\pi\left(\frac{2}{3g}\right)^{1/2} = \boxed{1.6\text{ s}}$

b) From problem 2, the frequency of a simple pendulum is

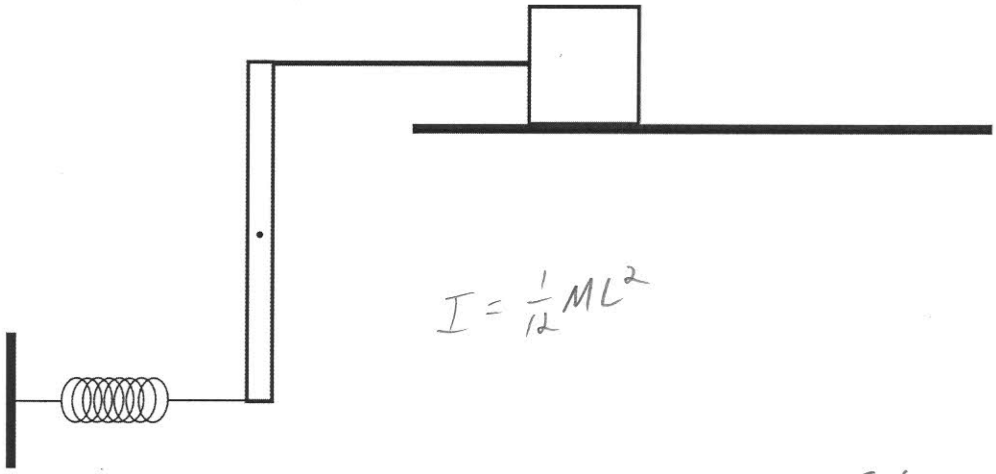
$\omega = \sqrt{g/L}$ so $T = \frac{2\pi}{\omega} \Rightarrow T = 2\pi\sqrt{L/g}$

We want $T_m = T_s$ so: $\cancel{2\pi}\sqrt{L/g} = \cancel{2\pi}\left(\frac{2}{3g}\right)^{1/2} \Rightarrow \frac{L}{\cancel{g}} = \frac{2}{3}\frac{1}{\cancel{g}}$

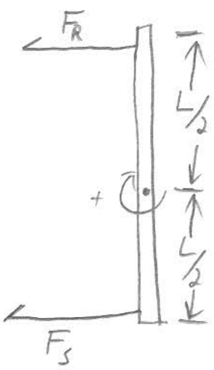
meter stick   simple

$$\boxed{L = 2/3\text{ m}}$$

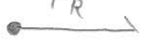
A block of mass M resting on a frictionless surface is attached to a stiff rod of negligible mass. The other end of the rod is attached to the top of a thin bar of length l mass M that is allowed to rotate about its center. The bottom of a bar is attached to a light spring of spring constant k. The spring is relaxed when the bar is vertical. Find the frequency of small oscillations.



$I = \frac{1}{12}ML^2$



Ooh... Those pesky reaction forces...



NSL

$\frac{L}{2}F_S - \frac{L}{2}F_R = I\alpha$ → Torque

$F_R = Ma$ → Force

$\Rightarrow F_s - Ma = 2\frac{I}{L}\alpha$

$\Rightarrow -kx = \frac{\cancel{2}}{\cancel{L}}\cdot\frac{1}{\cancel{12}\,6}ML^{\cancel{2}}\alpha + Ma$

Let's go with Translation...

$-kx = \frac{1}{\cancel{6}\,3}ML\frac{a}{\cancel{L/2}} + Ma \Rightarrow -kx = \frac{1}{3}Ma + Ma$

$\Rightarrow -kx = \frac{4}{3}Ma \Rightarrow \frac{d^2x}{dt^2} = -\frac{3}{4}\frac{k}{M}x$

$$\boxed{\omega = \left(\frac{3}{4}\frac{k}{M}\right)^{1/2}}$$