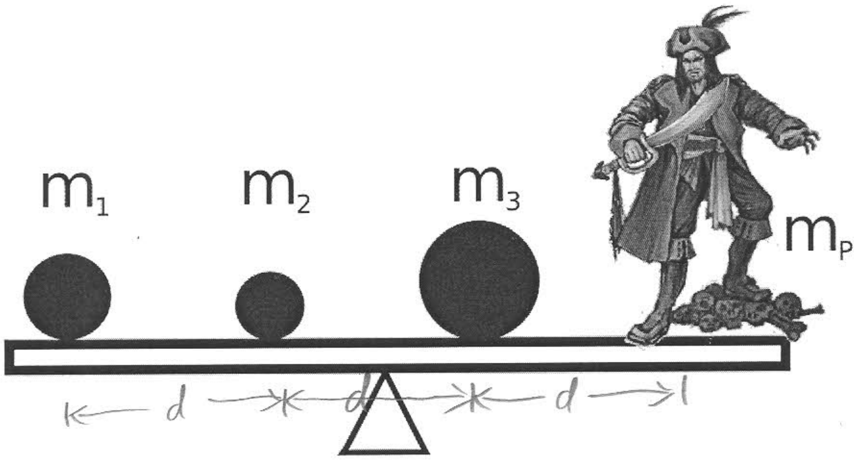
For reasons that nobody can explain, the Evil Pirate wants to stand on one end of a Plank of Negligible Mass (it's presumably used for walking) and be perfectly balanced by balls from the *Horrible Pendulum of Doom*. He's a little bit OCD, so he insists that all of the objects on the plank must be separated by the same distance *d*.



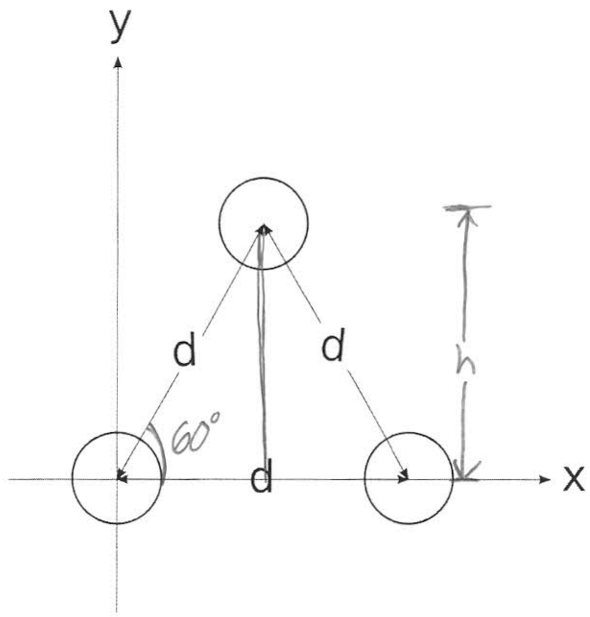
You can avoid walking the Plank of Negligible Mass if you can tell him where to put his fulcrum so that he can stand happily balanced and calm.
HINT: Put the fulcrum at the center of mass.

I'll put m_1 at $x = 0$ and calculate the center of mass using the Mass Weighted Average Position.

$$x_{cm} = \frac{\sum m_i x_i}{\sum m_i} = \frac{m_1(0) + m_2(d) + m_3 2d + m_p 3d}{m_1 + m_2 + m_3 + m_p}$$

$$\boxed{x_{cm} = d\,\frac{m_2 + 2m_3 + 3m_p}{m_1 + m_2 + m_3 + m_p}}$$

Three balls of equal mass form an equilateral triangle. Find the coordinates of the center of mass of the system.



2D problem, calculate x_{cm} and y_{cm} separately.

$$x_{cm} = \frac{\sum m_i x_i}{\sum m_i} = \frac{0 \cdot m + \frac{d}{2} m + dm}{3m} = \frac{d\cancel{m}}{3\cancel{m}}\left(\frac{1}{2} + 1\right) = \frac{d}{\cancel{3}} \cdot \frac{\cancel{3}}{2} = \boxed{\frac{d}{2}}$$

$$y_{cm} = \frac{\sum m_i y_i}{\sum m_i} = \frac{0 \cdot m + 0 \cdot m + hm}{3m}$$

$h = d\,\mathrm{SIN}\,60$

$\Rightarrow h = d\frac{\sqrt{3}}{2}$

$$y_{cm} = \frac{d\frac{\sqrt{3}}{2}\cancel{m}}{3\cancel{m}} = d\,\frac{\sqrt{3}}{2 \cdot 3} = \boxed{\frac{d}{2\sqrt{3}}}$$

We are going to calculate the location of the center of mass of a thin uniform rod of mass M and length L. The integral form of the center of mass equation is:

$$l_{cm} = \frac{\int_{l_0}^{L} l\,dm}{\int_{l_0}^{L} dm}$$

The variable of integration is l, but the differential is dm. We need to rewrite dm in terms of l. In other words, we need to perform a *Change of Variables*. The following steps will walk us through it.

a. You're in the hardware store and you notice that chain is on sale for $1.00 per pound. The chain you want is 3 pounds per foot and you want 10 feet. How much do you have to spend?

One to one dollar to pound correspondance, so lb and $ are the same

$$\left(3\frac{lb}{\cancel{ft}}\right)\cdot 10\,\cancel{ft} = \underline{30\,lb} \quad \text{or} \quad \underline{\$30}$$

b. You see that your neighbor is walking out the door with 50 feet of chain. He says gleefully that he paid only 25 dollars for it. What is the *linear density* (mass per unit length) of his chain ?

$$\$25 = 25\,lb$$

$$\lambda = \frac{25\,lb}{50\,Ft} = \boxed{\frac{1}{2}\frac{lb}{Ft}}$$

linear density

c. What would you pay for 30 feet of the chain that your neighbor bought?

linear density times length gives mass.

$$\frac{1}{2}\frac{lb}{\cancel{ft}}\cdot 30\,\cancel{ft} = 15\,lb \quad \text{or} \quad \underline{\$15}$$

d. We generally assign linear density the variable λ. What is the linear density (λ) of a uniform rod of mass M and length L? (refer to step b for assistance)

Linear density is mass over distance

So: $\boxed{\lambda = \frac{M}{L}}$

d. Now, write an equation for the mass m of a piece of this rod in terms of the total mass M, the total length L, and the length of the piece l. (refer to step c for assistance).

linear density times length equals mass...

$$m = \lambda l \Rightarrow \boxed{m = \frac{M}{L} l}$$

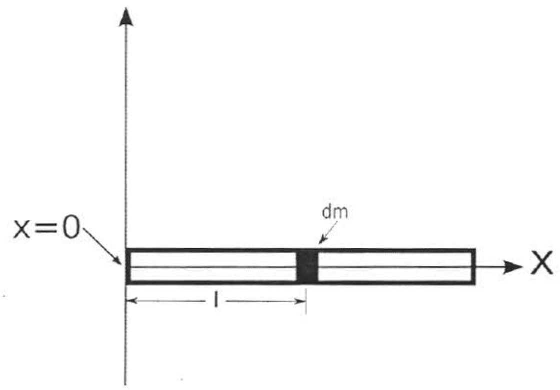
e. Calculate the derivative $\frac{dm}{dl}$ of the equation above. Now, what is dm in terms of M, L, and dl?

$$\frac{dm}{dl} = \frac{d}{dl}\lambda l = \lambda$$

This is REALLY important

$$\frac{dm}{dl} = \lambda \Rightarrow \boxed{dm = \lambda\, dl} \Rightarrow \boxed{dm = \frac{M}{L} dl}$$

f. Taking the lower integral first, use the picture below and what we discovered in part e to calculate the total mass. Hopefully you get M.



$$\int_0^L dm = \int_0^L \frac{M}{L} dl = \frac{M}{L}\int_0^L dl$$

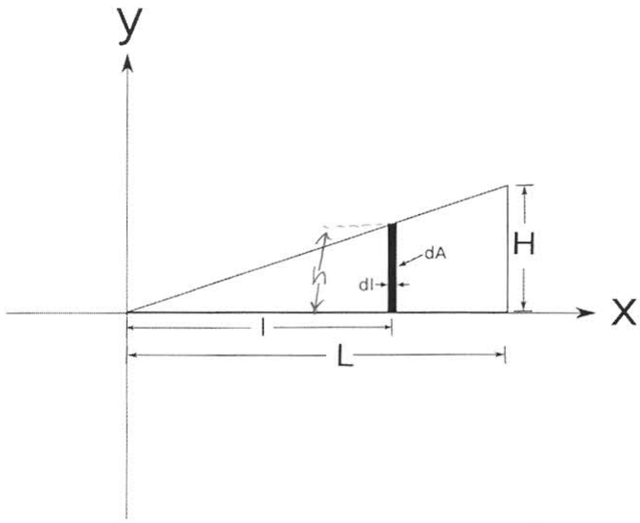
$$= \frac{M}{\cancel{L}}\cancel{L} = \boxed{M}$$

g. Now calculate the the position of the center of mass. Hopefully, you get L/2.

$$l_{cm} = \frac{\int l\, dm}{\int dm} = \frac{1}{\cancel{M}}\int_0^L l \frac{\cancel{M}}{L} dl = \frac{1}{L}\int_0^L l\, dl = \frac{1}{L}\left(\tfrac{1}{2} l^2\right)\Big|_0^L = \frac{1 L^{\cancel{2}}}{2 \cancel{L}}$$

$$\boxed{l_{cm} = \frac{L}{2}}$$

Calculate the center of mass of a triangular chunk of aluminum of mass *M*, length *l*, and height *h*.



a) Write an expression for *dm* in terms of *dl* similarly to what we did for the uniform rod.

Now, we consider a slice of the triangle with mass dm and area dA.

The slice has height h and width dl so: $\boxed{dA = h\,dl}$

Linear density (λ) times length gives mass so...
Surface density (σ) times area gives mass

$\boxed{dm = \sigma\,dA} \Rightarrow dm = \sigma h\,dl$

σ has units of mass over area so:

$$\sigma = \frac{M}{\frac{1}{2}LH} = \frac{2M}{LH} \Rightarrow \boxed{dm = \frac{2M}{LH} h\,dl}$$

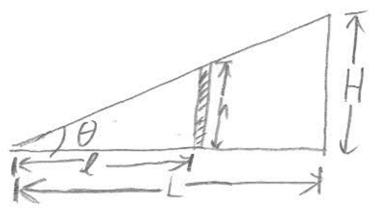
continued ↓

Systems Set 1 - P5 continued

b) calculate x_{cm}

$$x_{cm} = \frac{1}{M}\int_0^L l\,dm = \frac{1}{\cancel{M}}\int_0^L l\,\frac{2\cancel{M}}{LH}\,h\,dl = \frac{2}{LH}\int_0^L l h\,dl$$

h depends on l so we have to rewrite it:



$\tan\theta = \frac{H}{L}$, $\tan\theta = \frac{h}{l}$

so: $\frac{H}{L} = \frac{h}{l} \Rightarrow \boxed{h = l\frac{H}{L}}$

$$x_{cm} = \frac{2}{L\cancel{H}}\int_0^L l\cdot l\,\frac{\cancel{H}}{L}\,dl = \frac{2}{L^2}\int_0^L l^2\,dl = \frac{2}{L^2}\left(\frac{1}{3}l^3\right)\Big|_0^L = \frac{2}{\cancel{L^2}}\cdot\frac{1}{3}L^{\cancel{3}}$$

$$\boxed{x_{cm} = \frac{2}{3}L}$$

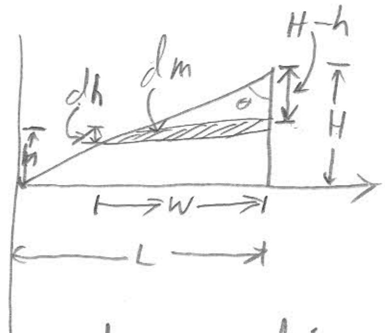
c) Calculate y_{cm}

$$y_{cm} = \frac{1}{M}\int_0^H h\,dm = \frac{1}{\cancel{M}}\int_0^H h\,\frac{2\cancel{M}}{HL}\,w\,dh$$

$$= \frac{2}{H\cancel{L}}\int_0^H h\,\frac{\cancel{L}}{H}(H-h)\,dh$$

$$= \frac{2}{H^2}\int_0^H (hH - h^2)\,dh = \frac{2}{\cancel{H^2}}\left(\frac{1}{2}H^{\cancel{3}} - \frac{1}{3}H^{\cancel{3}}\right)$$

$$= H\left(1 - \tfrac{2}{3}\right) = \boxed{\tfrac{1}{3}H}$$



$dm = \sigma\,dA$

$$\boxed{dm = \frac{2M}{HL}\,w\,dh}$$

and $\frac{w}{H-h} = \frac{L}{H}$

$$\Rightarrow \boxed{w = \frac{L}{H}(H-h)}$$

A thin rod of length L has a linear density of $\lambda = \lambda_0\left[2\frac{l^2}{L^2} + \frac{1}{3}\right]$.

where l is the distance from one end of the rod and λ_0 is a constant with units of mass per unit length.

a) Calculate the total mass of the rod.
b) Calculate the center of mass of the rod.

$$M = \int_0^L dm = \int_0^L \lambda\, dl = \int_0^L \lambda_0\left[2\frac{l^2}{L^2} + \frac{1}{3}\right] dl$$

$$= \lambda_0\left(\frac{2}{3}\frac{L^{\cancel{3}}}{\cancel{L^2}} + \frac{1}{3}L\right) = \boxed{\lambda_0 L}$$

$$b)\ l_{cm} = \frac{1}{M}\int_0^L l\, dm = \frac{1}{\lambda_0 L}\int_0^L l\lambda\, dl = \frac{1}{\cancel{\lambda_0} L}\int_0^L l\cancel{\lambda_0}\left[\frac{2l^2}{L^2} + \frac{1}{3}\right]$$

$$= \frac{1}{L}\int_0^L \left(\frac{2l^3}{L^2} + \frac{2l}{3}\right) dl = \frac{1}{\cancel{L}}\left(\frac{2}{4}\frac{L^{\cancel{4}\,2}}{\cancel{L^2}} + \frac{1}{2}\frac{L^{\cancel{2}}}{3}\right) = L\left(\frac{1}{2} + \frac{1}{6}\right)$$

$$\boxed{l_{cm} = \frac{2}{3}L}$$