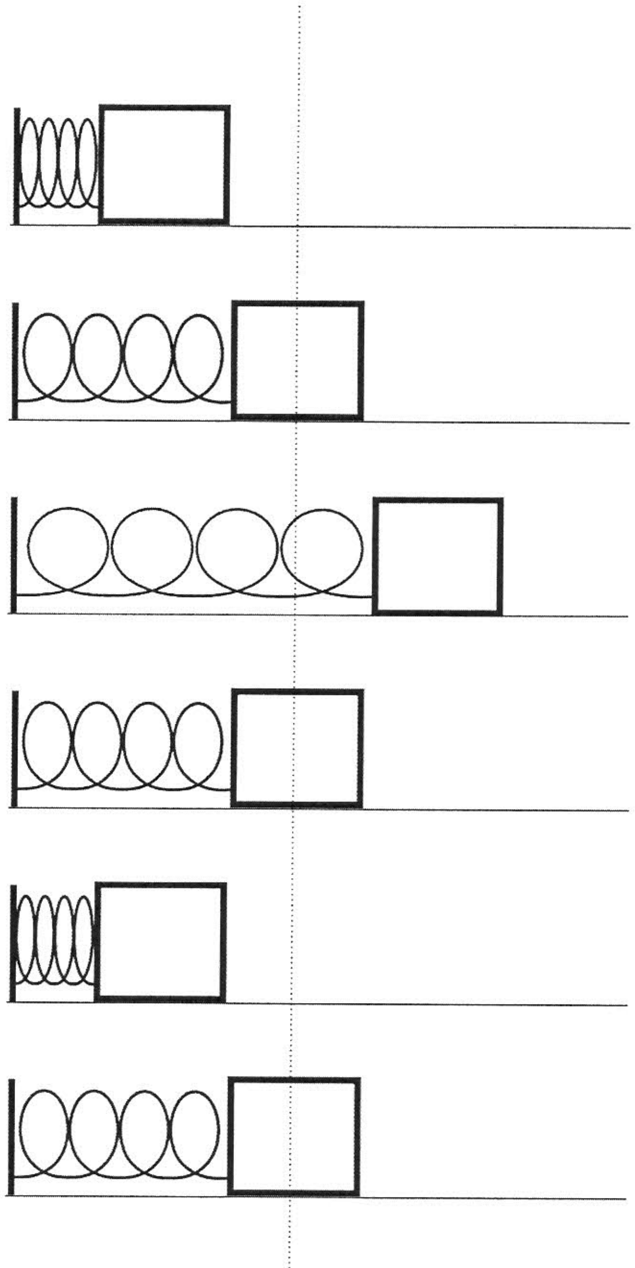
Oscillation – Set 1

Each row in the table below represents a snapshot of a mass attached to a spring. Assume that the mass starts from rest in the first row. In the second row, it is passing through x=0. In the third row, it has reached its maximum extension. In the fifth row, it has reached its maximum compression. In the cells below, mark an arrow indicating the direction of the associated force, acceleration, velocity, and position vectors for each row. If the magnitude is zero, put a zero in the cell.

F	a	v	x
→	→	0	←
0	0	→	0
←	←	0	→
0	0	←	0
→	→	0	←
0	0	→	0



x=0

The general form of the differential equation describing a simple harmonic oscillator is:

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

The function *x(t)* that satisfies the above equation is:

$$x(t) = A\cos(\omega t + \varphi)$$

where A, ω, and φ are constants.

a) Derive expressions for the *velocity* and *acceleration* of a simple harmonic oscillator.
HINT: What are the definitions of velocity and acceleration?

$v = \frac{dx}{dt} = \frac{d}{dt}(A\cos(\omega t + \phi))$ $a = \frac{dv}{dt} = \frac{d}{dt}(-\omega A\sin(\omega t + \phi))$

$\boxed{v = -\omega A \sin(\omega t + \phi)}$ $\boxed{a = -\omega^2 A\cos(\omega t + \phi)}$

b) Does our equation for *x(t)* actually satisfy the differential equation?

Yes. After 2 dervatives, we get $x(t)$ back as well as $\underline{-\omega^2}$.

c) Why can't we apply the kinematics equations to a simple harmonic oscillator?

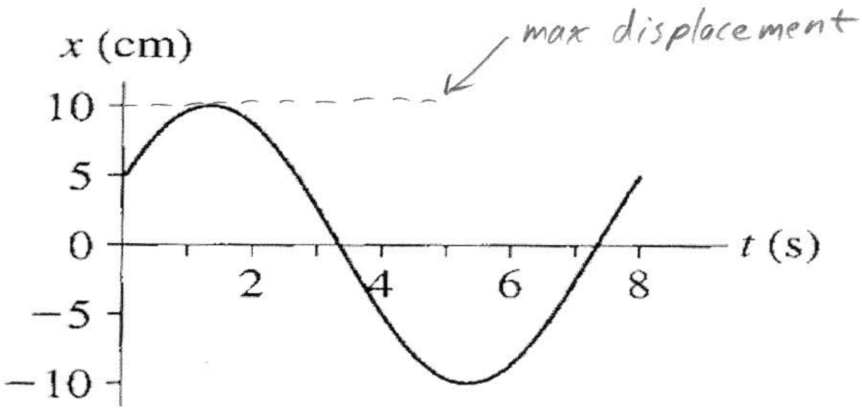
Neither F nor a is constant.

Oscillation – Set 1

The figure below is a position versus time graph of a particle in simple harmonic motion. Assume that its position as a function of time is given by

$$x(t) = A\cos(\omega t + \varphi)$$

where A, ω and φ are constants.



a) What is the maximum displacement (amplitude) of the particle?

10cm

b) Which constant in the above equation gives the maximum displacement, or amplitude, of the oscillations? *(HINT: What's the maximum possible value of cosine?)*

max value of cosine is +1.

$x_{max} = A \cdot 1 = \boxed{A}$

c) What is the value *x(0)* (ie. *x* when *t* = 0)?

at $t=0$, $x = 5$cm. so: $x(0) = 5$

d) Given your answer to part c, solve $x(0) = A\cos(\omega t + \varphi)$ for φ (the phase constant) when t = 0?

$x(0) = A\cos(\omega t + \phi)$

$5 = 10\cos(0 + \phi) \Rightarrow \cos\phi = \frac{1}{2}$

$\boxed{\phi = \cos^{-1}(\frac{1}{2})}$

e) What is the period (T) of the oscillations?

Period is the time required for one cycle.
According to the graph, $T = 8$ sec.

f) What are the units of ω? (*HINT: What are the units of the input to the cosine function?*)

Cosine takes an angle, (let's use radians)
so $(\omega t + \phi)$ must have units of radians.
$\Rightarrow$, $\boxed{\omega \text{ must be rad/sec}}$

g) What is the mathematical relationship between ω and T?

$T\left(\frac{sec}{cycle}\right)$ and there are $2\pi\left(\frac{rad}{cycle}\right)$ so $\boxed{T = \frac{2\pi}{\omega}}$

$\omega\left(\frac{rad}{sec}\right)$ $\quad T = \dfrac{2\pi\left(\cancel{rad}/cycle\right)}{\omega\left(\cancel{rad}/sec\right)} = \dfrac{2\pi}{\omega}\left(\frac{sec}{cycle}\right)$

h) Good! Now calculate the numerical value of ω.

$$\omega = \frac{2\pi}{8} = \boxed{\frac{\pi}{4}\ \text{rad/s}}$$

d) What is the maximum velocity of the particle?
(HINT: What's the maximum possible value of sine?)

$v = -\omega A \sin(\omega t + \phi)$

when $\sin = -1$, $v = \omega A = \frac{\pi}{4} \cdot 10 = \boxed{\frac{5}{2}\pi}$

e) What is the maximum acceleration of the particle?
(HINT: What's the maximum possible value of cosine?)

$a = -\omega^2 A \cos(\omega t + \phi)$

when $\cos = -1$, $a = \omega^2 A = \frac{\pi^2}{16} \cdot 10 = \boxed{\frac{5}{8}\pi^2}$

Oscillation – Set 1

A block with a mass of m = 2.00 kg is attached to a spring with a spring constant k = 100 N/m. When t = 1.00 s, the position and velocity of the block are x(1s) = 0.129 m and v(1s) = 3.415 m/s.

a) Find the angular frequency, ω, of the oscillator.

b) Find the phase constant.

c) Find the amplitude

d) What was the position of the block at t = 0.00 s?

Step 1 – Use NSL to find the oscillator frequency.

FBD

NSL

$$\Sigma F = ma$$

$$F_s = m\frac{d^2x}{dt^2}, \quad F_s = -kx$$

$$\Rightarrow -kx = m\frac{d^2x}{dt^2}$$

$$\Rightarrow \boxed{\frac{d^2x}{dt^2} = -\frac{k}{m}x} \Rightarrow$$ Simple Harmonic Oscillator!

Now, to get the oscillator frequency, we can plug our general solution into the above equation.

General solution: $x(t) = A\cos(\omega t + \phi)$

$$\Rightarrow \frac{d^2}{dt^2}\left[A\cos(\omega t + \phi)\right] = -\frac{k}{m}A\cos(\omega t + \phi)$$

$$\Rightarrow -\omega^2 \cancel{A\cos(\omega t + \phi)} = -\frac{k}{m}\cancel{A\cos(\omega t + \phi)}$$

$$\Rightarrow -\omega^2 = -\frac{k}{m} \Rightarrow \boxed{\omega = \sqrt{\frac{k}{m}}} \Rightarrow \omega = \sqrt{50} = \underline{7.07}$$

Oscillation Set 1, P4 continued

Step ② - Use Physics to find the initial conditions

In this case, they're given in the problem statement.

at $t = 1.0s$, $\boxed{x(1) = 0.129m}$, $\boxed{v(1) = 3.415\, m/s}$

Step ③ - Use the SHO general solution to solve the rest of the problem.

General Solution:

$$x(t) = A\cos(\omega t + \phi)$$
$$v(t) = -\omega A \sin(\omega t + \phi)$$

Apply our initial conditions

$$x(1) = A\cos(\omega(1) + \phi)$$
$$v(1) = -\omega A \sin(\omega(1) + \phi)$$

Now we can divide to eliminate A and solve for ϕ

$$\frac{v(1)}{x(1)} = \frac{-\omega A \sin(\omega + \phi)}{A\cos(\omega + \phi)} \Rightarrow \frac{v(1)}{x(1)} = -\omega \tan(\omega + \phi)$$

$$\Rightarrow \tan(\omega + \phi) = -\frac{v(1)}{\omega x(1)}$$

$$\Rightarrow \omega + \phi = \tan^{-1}\left(\frac{-v(1)}{\omega x(1)}\right)$$

$$\Rightarrow \boxed{\phi = \tan^{-1}\left(\frac{-v(1)}{\omega x(1)}\right) - \omega}$$

$$\Rightarrow \phi = \tan^{-1}\left(\frac{-3.415}{(7.07)(0.129)}\right) - 7.07 = \boxed{-8.38 \text{ radians}}$$

Oscillations Set 1, P4 continued

Plug our answer for ϕ back into the $x(1)$ eq. and solve for A:

$$x(1) = A\cos(\omega + \phi)$$

$$\Rightarrow \boxed{A = \frac{x(1)}{\cos(\omega + \phi)}}$$

$$\Rightarrow \boxed{A = \frac{0.129}{\cos(7.07 - 8.38)} = 0.5\,m}$$

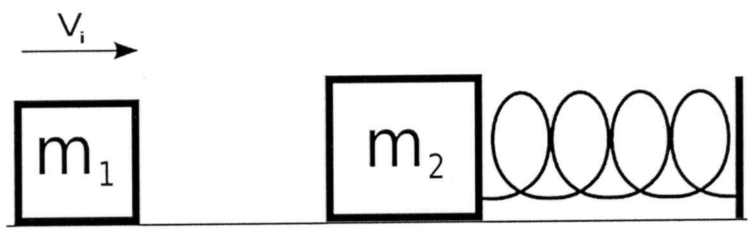
Finally, Put it all together to find $x(0)$

$$x(0) = 0.5\cos(\omega \cdot (0) - 8.38)$$

$$\boxed{x(0) = -0.25\,m}$$

Oscillation – Set 1

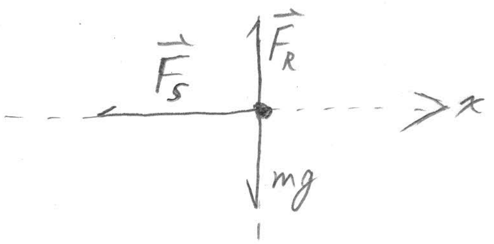
A block with a mass of m_1 = 10 kg is moving to the right with a velocity V_i. It collides and sticks to a block with a mass of m_2 = 15 kg. The second mass is attached to a spring with spring constant k=3 N/m. Before the collision, the spring is at rest in it's equilibrium position.



a) What is the frequency, ~~k~~ ω, of the resulting oscillator after the collision?
b) Assuming that the moment of collision is t=0, find the phase constant of the oscillator?
c) If the resulting amplitude of the oscillator is A = 3 m, what was the initial velocity of m_1?

Step 1 - Use NSL to find ω, the oscillator frequency
The oscillator starts after the collision so:

FBD



NSL

$$\sum \vec{F} = m\frac{d^2x}{dt^2}$$

$$-kx = m_T\frac{d^2x}{dt^2} \Rightarrow \boxed{\frac{d^2x}{dt^2} = -\frac{k}{m}x} \quad (1)$$

General solution is: $\boxed{x(t) = A\cos(\omega t + \phi)}$ (2)

so, combine (1) and (2)

$$\frac{d^2}{dt^2}\left[A\cos(\omega t + \phi)\right] = -\frac{k}{m_T}\left(A\cos(\omega t + \phi)\right)$$

$$\Rightarrow -\omega^2 \cancel{A\cos(\omega t+\phi)} = -\frac{k}{m_T}\left(\cancel{A\cos(\omega t+\phi)}\right)$$

$$\Rightarrow \boxed{\omega = \sqrt{\frac{k}{m}}} \Rightarrow \omega = \left(\frac{3}{(10+15)}\right)^{1/2} \Rightarrow \boxed{\omega = 0.35 \text{ rad/s}}$$

Oscillation Problems Set 1, P5 continued

Step ② - Use physics to find the initial conditions

Let: $t=0$ be the moment of collision.

$\boxed{x(t=0)=0}$, starting position is at equilibrium.

Let's find $v(t=0)=v_F$. It's an inelastic collision

$$P_I = P_F$$

$$m_1 v_{1I} = (m_1 + m_2) v_F$$

$$\Rightarrow \boxed{v_F = \frac{m_1}{m_1+m_2} v_{1I}}$$

Step ③ - Use the general solution of a SHO to solve the problem.

Part b asks for the phase constant.

Let's find it.

In general:

$$x(t) = A\cos(\omega t + \phi)$$

$$v(t) = -\omega A \sin(\omega t + \phi)$$

$$\Rightarrow 0 = A\cos(\phi) \Rightarrow \cos(\phi) = 0$$

True when $\phi = \pi/2, 3/2\pi$

continued ↓

Oscillation Problems Set 1, P5 continued

But we need $v(0)$ to be positive:

$v(0) = -\omega A \sin(\phi)$, $\sin\left(\frac{\pi}{2}\right) = 1$, $\sin\left(\frac{3\pi}{2}\right) = -1$

so, $\boxed{\phi = 3\pi/2}$

Now, we'll use the velocity equation to find v_{1I}

$v(0) = v_F = -\omega A \sin(\phi)$, $\phi = 3\pi/2$ so $\sin(\phi) = -1$

$$\Rightarrow \frac{m_1}{m_1 + m_2} v_{1I} = \omega A$$

$$\Rightarrow \boxed{v_{1I} = \frac{m_1 + m_2}{m_1} \omega A}$$

$$v_{1I} = \frac{10 + 15}{10}(0.35)(3)$$

$$\boxed{v_{1I} = 2.6 \text{ m/s}}$$

You are given the position and velocity of a simple harmonic oscillator (SHO) at some time t:

$$x(t) = x_0 \text{ and } v(t) = v_0$$

a) Starting with the equations for position and velocity that you derived in question 2, find an expression for the amplitude, A, of a SHO.

At time t, $x = x_0$ and $v = v_0$

① $x_0 = A\cos(\omega t + \phi)$, $v_0 = -\omega A \sin(\omega t + \phi)$

$$\Rightarrow \frac{v_0}{\omega} = -A\sin(\omega t + \phi) \quad ②$$

Square ① and ② and add them together

$$\Rightarrow x_0^2 + \frac{v_0^2}{\omega^2} = A^2\cos^2(\omega t + \phi) + A^2\sin^2(\omega t + \phi)$$

$$= A^2\underbrace{(\cos^2(\omega t + \phi) + \sin^2(\omega t + \phi))}$$ ← This is 1

$$\Rightarrow \boxed{A = \left(x_0^2 + v_0^2/\omega^2\right)^{1/2}}$$

b) Starting with the equations for position and velocity that you derived in question 2, find an expression for the phase angle, φ, of a SHO.

$x_0 = A\cos(\omega t + \phi)$, $v_0 = -\omega A \sin(\omega t + \phi)$

$$\frac{v_0}{x_0} = \frac{-\omega \cancel{A} \sin(\omega t + \phi)}{\cancel{A}\cos(\omega t + \phi)} = -\omega \tan(\omega t + \phi)$$

$$\tan(\omega t + \phi) = -\frac{v_0}{\omega x_0}$$

$$\Rightarrow \boxed{\phi = \tan^{-1}\left(-\frac{v_0}{\omega x_0}\right) - \omega t}$$