

Oscillation – Set 2

1

A block of mass m is attached to a spring with spring constant k and allowed to move on a frictionless surface.



1) Draw a Free Body Diagram for this system.



2) Let the equilibrium position of the spring be $x=0$, write **Newton's Second Law**, and solve it for the acceleration of the system.

$$F = ma$$
$$-kx = ma \Rightarrow a = -\frac{k}{m}x \Rightarrow \boxed{\frac{d^2x}{dt^2} = -\frac{k}{m}x}$$

3) Show that the **Simple Harmonic Oscillator** equation satisfies the differential equation above using substitution.

$$\text{SHO general solution: } x(t) = A \cos(\omega t + \phi)$$

$$\text{so: } \frac{d^2}{dt^2} [A \cos(\omega t + \phi)] = -\frac{k}{m} A \cos(\omega t + \phi)$$

$$\Rightarrow +\omega^2 A \cos(\omega t + \phi) = +\frac{k}{m} A \cos(\omega t + \phi) \Rightarrow \boxed{\omega = \sqrt{\frac{k}{m}}}$$

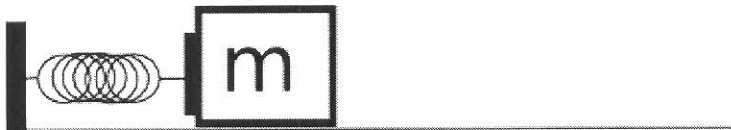
4) What did you find for the angular frequency of the oscillator?

$$\omega = \sqrt{\frac{k}{m}}$$

Oscillation – Set

2

A block of mass m is attached to a spring with spring constant k and allowed to move on a frictionless surface.



At $t=0$, the block is at $x=0$ (the spring's equilibrium point), and is moving to the right with a velocity $v=v_0$.

Find the amplitude A and the phase angle ϕ .

Initial conditions

$$x(0) = 0$$

$$v(0) = v_0$$

$$\text{so: } x(t) = A \cos(\omega t + \phi)$$

$$\Rightarrow 0 = A \cos(\phi)$$

$$\Rightarrow \phi = \cos^{-1}(0) \Rightarrow \phi = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}$$

For + initial v

$$\text{and } v(t) = -\omega A \sin(\omega t + \phi)$$

$$\Rightarrow v_0 = -\omega A \sin\left(\frac{3\pi}{2}\right) \Rightarrow A = \frac{v_0}{\omega}$$

What happens to A as v_0 is increased?

A increases

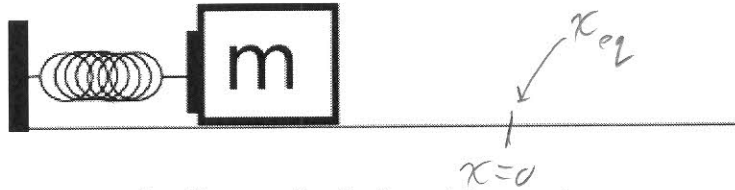
What happens to ϕ as v_0 is increased?

no change

Oscillation – Set

3

A block of mass m is attached to a spring with spring constant k and allowed to move on a frictionless surface.



At $t=0$, the block is compressed a distance d and released from rest.

Find the amplitude A and the phase angle ϕ .

Given
 $x(0) = d$
 $v(0) = 0$

$$x(t) = A \cos(\omega t + \phi) \Rightarrow d = A \cos(\phi) \quad (1)$$

$$v(t) = -\omega A \sin(\omega t + \phi) \Rightarrow 0 = -\omega A \sin(\phi) \quad (2)$$

From (2): $\sin(\phi) = 0 \Rightarrow \phi = 0$ or π neg. init. displacement

From (1): $d = A \cos(\pi) \Rightarrow A = -d$

What happens to A as d is increased?

A increases

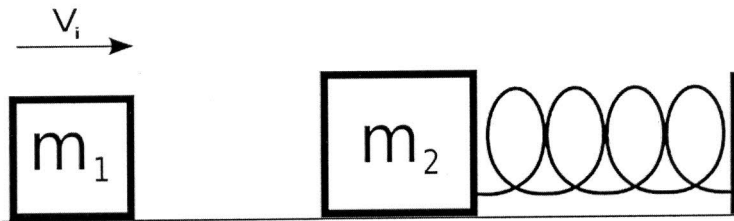
What happens to ϕ as d is increased?

No change

Why is ϕ different than it was in the last problem?

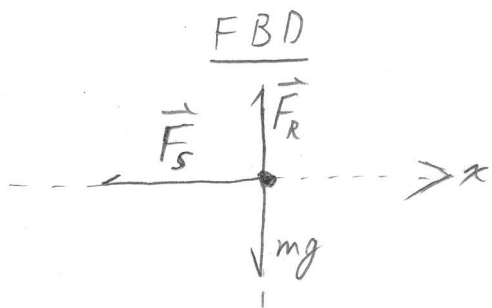
$t=0$ is at different points in the oscillator cycle.

A block with a mass of $m_1 = 10$ kg is moving to the right with a velocity V_i . It collides and sticks to a block with a mass of $m_2 = 15$ kg. The second mass is attached to a spring with spring constant $k=3$ N/m. Before the collision, the spring is at rest in its equilibrium position.



- What is the frequency, ω , of the resulting oscillator after the collision?
- Assuming that the moment of collision is $t=0$, find the phase constant of the oscillator?
- If the resulting amplitude of the oscillator is $A = 3$ m, what was the initial velocity of m_1 ?

Step 1 - Use NSL to find ω , the oscillator frequency
The oscillator starts after the collision so:



$$\begin{aligned} \text{NSL} \\ \sum \vec{F} &= m \frac{d^2x}{dt^2} \\ -kx &= m \frac{d^2x}{dt^2} \Rightarrow \boxed{\frac{d^2x}{dt^2} = -\frac{k}{m}x} \quad \text{①} \end{aligned}$$

General solution is: $\boxed{x(t) = A \cos(\omega t + \phi)}$ ②

so, combine ① and ②

$$\begin{aligned} \frac{d^2}{dt^2} [A \cos(\omega t + \phi)] &= -\frac{k}{m} (A \cos(\omega t + \phi)) \\ \Rightarrow -\omega^2 A \cos(\omega t + \phi) &= -\frac{k}{m} (A \cos(\omega t + \phi)) \\ \Rightarrow \boxed{\omega = \sqrt{\frac{k}{m}}} &\Rightarrow \omega = \left(\frac{3}{10+15}\right)^{1/2} \Rightarrow \boxed{\omega = 0.35 \text{ rad/s}} \end{aligned}$$

Oscillation Problems Set 1, P5 continued

Step ② - Use Physics to find the initial conditions

Let: $t=0$ be the moment of collision.

$x(t=0) = 0$, starting position is at equilibrium.

Let's find $v(t=0) = v_F$. It's an inelastic collision

$$p_I = p_F$$
$$m_1 v_{1I} = (m_1 + m_2) v_F$$

$$\Rightarrow v_F = \frac{m_1}{m_1 + m_2} v_{1I}$$

Step ③ - Use the general solution of a SHO to solve the problem.

Part b asks for the phase constant.

Let's find it.

In general:

$$x(t) = A \cos(\omega t + \phi)$$

$$v(t) = -\omega A \sin(\omega t + \phi)$$

$$\Rightarrow 0 = A \cos(\phi) \Rightarrow \cos(\phi) = 0$$

True when $\phi = \frac{\pi}{2}, \frac{3}{2}\pi$

continued



Oscillation Problems Set 1, P5 continued

But we need $v(0)$ to be positive:

$$v(0) = -\omega A \sin(\phi), \quad \sin\left(\frac{\pi}{2}\right) = 1, \quad \sin\left(\frac{3\pi}{2}\right) = -1$$

$$\text{so, } \boxed{\phi = \frac{3\pi}{2}}$$

Now, we'll use the velocity equation to find v_{12}

$$v(0) = v_{12} = -\omega A \sin(\phi), \quad \phi = \frac{3\pi}{2} \text{ so } \sin(\phi) = -1$$

$$\Rightarrow \frac{m_1}{m_1 + m_2} v_{12} = \omega A$$

$$\Rightarrow \boxed{v_{12} = \frac{m_1 + m_2}{m_1} \omega A}$$

$$v_{12} = \frac{10 + 15}{10} (0.35)(3)$$

$$\boxed{v_{12} = 2.6 \text{ m/s}}$$

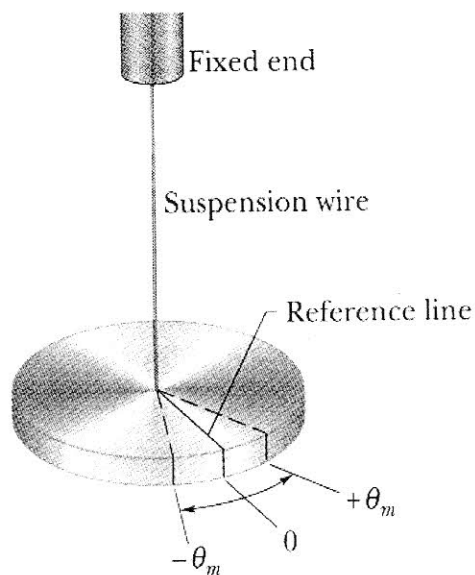
Oscillation – Set 2

1

The device in the picture below is known as a torsion pendulum. It is a flat disk attached to a length of stiff wire. When the wire is twisted, it responds by providing a torque on the disk, much the same way a spring provides a force when it is stretched. The torque provided by the wire is $T = -\kappa\theta$, where κ (greek letter kappa) is the torsion constant and θ is the angular displacement from equilibrium.

a) The moment of Inertia of the disk is $I = \frac{1}{2}MR^2$. Using the rotational version of Newton's Second Law, find the oscillator frequency of the torsion pendulum.

b) If a solid bar of length L , $I = \frac{1}{12}ML^2$, were suspended from the wire, what would the oscillator frequency be?



Let's solve the problem generally in terms of I , then plug in the different I for each object.

NSL for rotation

$$\Sigma T = I\alpha$$

$$-k\theta = I \frac{d^2\theta}{dt^2}$$

$$\Rightarrow \boxed{\frac{d^2\theta}{dt^2} = -\frac{k}{I}\theta} \text{ SHO!}$$

This is the simple harmonic oscillator equation, except in θ instead of x .

$$\omega = \left(\frac{k}{I}\right)^{1/2}$$

$$a) \omega = \left[\frac{2k}{MR^2}\right]^{1/2}$$

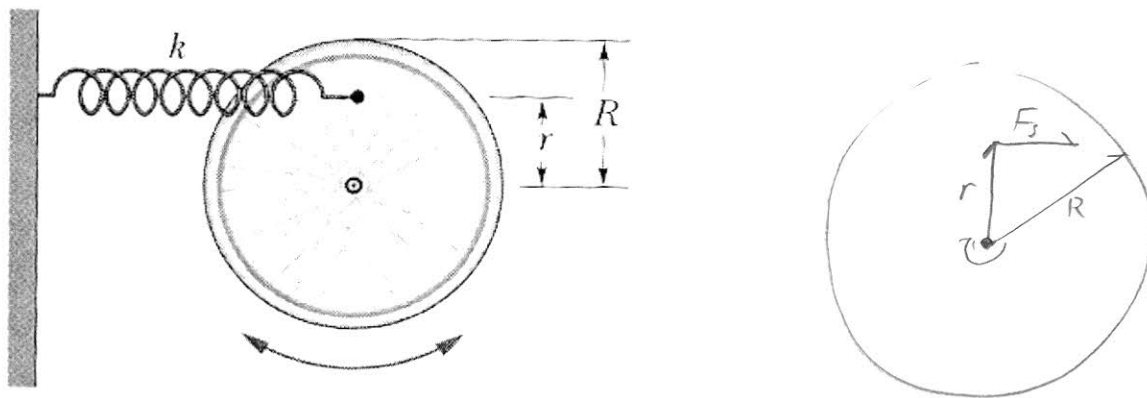
$$b) \omega = \left[\frac{12k}{ML^2}\right]^{1/2}$$

SAMPLE TEST 6

PHYS 111, FALL 2010, SECTION 1

A wheel is free to rotate about a fixed axle. A spring with a spring constant k is attached to one of its spokes at a distance r from the axle, as shown in the picture. Assume that the wheel is a hoop of mass m and radius R (the spokes have negligible mass).

- Using **Newton's Second Law**, find the angular frequency of small oscillations in terms of m , R , r and the spring constant k .
- Using **Energy techniques**, find the angular frequency of small oscillations in terms of m , R , r and the spring constant k .
- What is the angular frequency if $r = R$.
- What is the angular frequency if $r = 0$.



a) $\Sigma T = I\alpha$
 $F_s r = I\alpha$
 $-kx r = I\alpha$, $x = r\theta$
 $-kr\theta r = I\alpha$
 $\alpha = -\left[\frac{kr^2}{I}\right]\theta \Rightarrow \omega = \left[\frac{kr^2}{I}\right]^{1/2}$, $I = mR^2$
 $\Rightarrow \omega = \left[\frac{kr^2}{mR^2}\right]^{1/2}$

continued

Sample Test 6, P4 continued

b) $E_T = \frac{1}{2} k x^2 + \frac{1}{2} I \omega^2$ ← angular velocity!

$$\frac{dE_T}{dt} = \frac{1}{2} k \cancel{x} \frac{dx}{dt} + \frac{1}{2} I \cancel{\omega} \frac{d\omega}{dt} = 0$$

$$\Rightarrow k x v + I \omega \alpha = 0$$

$$v = r \omega \Rightarrow \omega = \frac{v}{r}$$

$$\Rightarrow k x \cancel{x} + m R^2 \frac{\cancel{x}}{r} \cdot \frac{a}{r} = 0$$

$$a = r \alpha \Rightarrow \alpha = \frac{a}{r}$$

$$\Rightarrow k x + m \frac{R^2}{r^2} a = 0$$

$$\Rightarrow a = - \left[\frac{k}{m} \frac{r^2}{R^2} \right] x \Rightarrow \omega = \left[\frac{k}{m} \frac{r^2}{R^2} \right]^{1/2}$$

c) if $R = r$, $\omega = \sqrt{\frac{k}{m}}$

d) if $r = 0$, $\omega = 0$

Oscillation – Set 1

4

A block with a mass of $m = 2.00$ kg is attached to a spring with a spring constant $k = 100$ N/m. When $t = 1.00$ s, the position and velocity of the block are $x(1s) = 0.129$ m and $v(1s) = 3.415$ m/s.

- Find the angular frequency, ω , of the oscillator.
- Find the phase constant.
- Find the amplitude
- What was the position of the block at $t = 0.00$ s?

Step 1 - Use NSL to find the oscillator frequency.

FBD



NSL

$$\sum F = ma$$

$$F_s = m \frac{d^2x}{dt^2}, \quad F_s = -kx$$

$$\Rightarrow -kx = m \frac{d^2x}{dt^2}$$

$$\Rightarrow \left[\frac{d^2x}{dt^2} = -\frac{k}{m}x \right] \Rightarrow \text{Simple Harmonic Oscillator!}$$

Now, to get the oscillator frequency, we can plug our general solution into the above equation.

$$\text{General solution: } x(t) = A \cos(\omega t + \phi)$$

$$\Rightarrow \frac{d^2}{dt^2} [A \cos(\omega t + \phi)] = -\frac{k}{m} A \cos(\omega t + \phi)$$

$$\Rightarrow -\omega^2 A \cos(\omega t + \phi) = -\frac{k}{m} A \cos(\omega t + \phi)$$

$$\Rightarrow -\omega^2 = -\frac{k}{m} \Rightarrow \left[\omega = \sqrt{\frac{k}{m}} \right] \Rightarrow \omega = \sqrt{50} = \underline{7.07}$$

Oscillation Set 1, P4 continued

Step 2 - Use physics to find the initial conditions

In this case, they're given in the problem statement.

$$\text{at } t = 1.0 \text{ s, } \boxed{x(1) = 0.129 \text{ m}} \quad \boxed{v(1) = 3.415 \text{ m/s}}$$

Step 3 - Use the SHO general solution to solve the rest of the problem.

General solution:

$$x(t) = A \cos(\omega t + \phi)$$

$$v(t) = -\omega A \sin(\omega t + \phi)$$

Apply our initial conditions

$$x(1) = A \cos(\omega(1) + \phi)$$

$$v(1) = -\omega A \sin(\omega(1) + \phi)$$

Now we can divide to eliminate A and solve for ϕ

$$\frac{v(1)}{x(1)} = \frac{-\omega A \sin(\omega + \phi)}{A \cos(\omega + \phi)} \Rightarrow \frac{v(1)}{x(1)} = -\omega \tan(\omega + \phi)$$

$$\Rightarrow \tan(\omega + \phi) = -\frac{v(1)}{\omega x(1)}$$

$$\Rightarrow \omega + \phi = \tan^{-1}\left(\frac{-v(1)}{\omega x(1)}\right)$$

$$\Rightarrow \boxed{\phi = \tan^{-1}\left(\frac{-v(1)}{\omega x(1)}\right) - \omega}$$

$$\Rightarrow \phi = \tan^{-1}\left(\frac{-3.415}{(7.07)(0.129)}\right) - 7.07 = \boxed{-8.38 \text{ radians}}$$

Oscillations Set 1, P4 continued

Plug our answer for ϕ back into the $x(t)$ eq. and solve for A :

$$x(t) = A \cos(\omega t + \phi)$$

$$\Rightarrow \boxed{A = \frac{x(t)}{\cos(\omega t + \phi)}}$$

$$\Rightarrow \boxed{A = \frac{0.129}{\cos(7.07 - 8.38)} = 0.5 \text{ m}}$$

Finally, Put it all together to find $x(0)$

$$x(0) = 0.5 \cos(\omega(0) - 8.38)$$

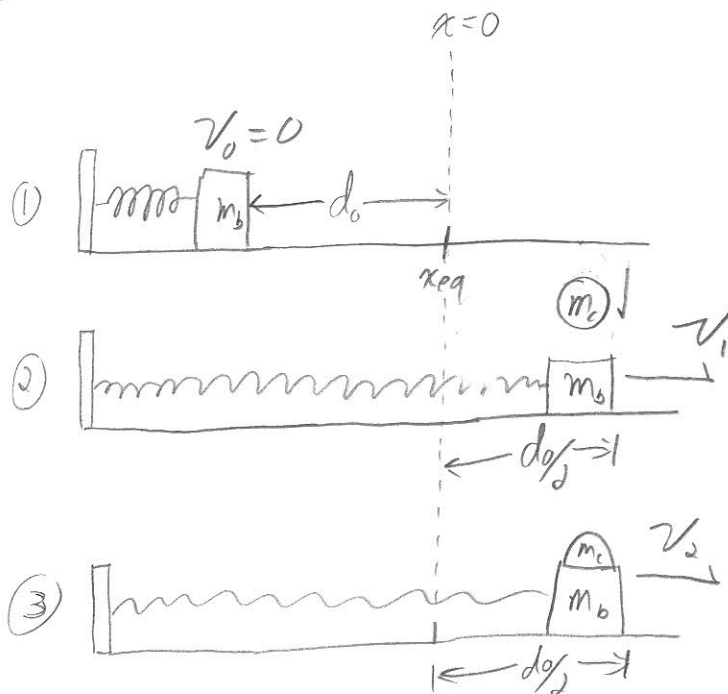
$$\boxed{x(0) = -0.25 \text{ m}}$$

Oscillation – Set 2

1) A block of mass m_b is attached to a spring of spring constant k where it is allowed to oscillate horizontally on a frictionless surface. The spring is compressed a distance d_0 from equilibrium and released. As the block passes $x=+d/2$, a wad of clay, m_c kg, falls from directly above and sticks to the block.

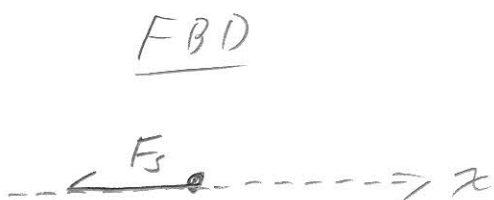
- a) What is the angular frequency of the block/clay oscillator system after the collision?
- b) What is the phase angle of the block/clay oscillator after the collision?
- c) What is the amplitude of the block/clay oscillator after the collision?

<u>Given</u>	<u>want</u>
m_b	ω
k	A
d_0	ϕ
$x_0 = \frac{d_0}{2}$	
m_c	



a) Write NSL to find ω .

Stage ③ is the oscillator in question, so write a FBD for this case



NSL

$$F = ma$$

$$-kx = ma \Rightarrow a = -\frac{k}{m}x$$

$$\Rightarrow \left| \frac{d^2x}{dt^2} = -\frac{k}{m}x \right| \textcircled{1}$$

continued
↓

Oscillation Set 2 Pt continued.

2

We know that the solution to the above eq. takes the form:

$$x(t) = A \cos(\omega t + \phi)$$

so, plug it into eq ①

$$\frac{d^2}{dt^2} [A \cos(\omega t + \phi)] = -\frac{k}{m} A \cos(\omega t + \phi)$$

$$\Rightarrow +\omega^2 A \cos(\omega t + \phi) = +\frac{k}{m} A \cos(\omega t + \phi)$$

$$\Rightarrow \omega = \sqrt{\frac{k}{m}}$$

$$\Rightarrow \boxed{\omega = \sqrt{\frac{k}{m_b + m_c}}}$$

b) Do Physics to Find $v(0)$ and $x(0)$

conserve energy from stage ① to stage ②

$$U_I = \frac{1}{2} k d_0^2 \quad K_I = 0$$

$$U_F = \frac{1}{2} k \left(\frac{d_0}{2}\right)^2 \quad K_F = \frac{1}{2} m_b v_1^2$$

$$\Rightarrow \cancel{\frac{1}{2}} k d_0^2 = \cancel{\frac{1}{2}} k \frac{1}{4} d_0^2 + \cancel{\frac{1}{2}} m_b v_1^2$$

$$\Rightarrow \frac{3}{4} k d_0^2 = m_b v_1^2 \Rightarrow \boxed{v_1 = \left[\frac{3}{4} \frac{k d_0^2}{m_b} \right]^{1/2}} \quad \text{②}$$

continued ↓

Oscillation set 2 P1 continued

③

conserve momentum From stage ② to stage ③

on the x -axis only

$$m_b v_1 = (m_b + m_c) v_2 \Rightarrow \boxed{v_2 = \frac{m_b}{m_b + m_c} v_1} \quad \text{③}$$

* Setting the moment of collision as $t=0$,

$$\boxed{\begin{aligned} v(0) &= v_2 = \frac{m_b}{m_b + m_c} \left[\frac{3}{4} \frac{k d_0^2}{m_b} \right]^{1/2} \\ x(0) &= \frac{d_0}{2} \end{aligned}}$$

* Now, use solutions for $x(t)$ and $v(t)$ to find ϕ

$$\begin{aligned} x(t) &= A \cos(\omega t + \phi) \Rightarrow x(0) = A \cos(\phi) \quad \text{④} \\ v(t) &= -\omega A \sin(\omega t + \phi) \Rightarrow v(0) = -\omega A \sin(\phi) \quad \text{⑤} \end{aligned}$$

$$\text{Divide: } \frac{v(0)}{x(0)} = \frac{-\omega A \sin(\phi)}{A \cos(\phi)}$$

$$\Rightarrow \tan \phi = -\frac{v(0)}{\omega x(0)}$$

$$\Rightarrow \boxed{\phi = \tan^{-1} \left[-\left(\frac{m_b k d_0^2}{(m_b + m_c)^2} \right)^{1/2} \frac{2}{d_0} \right]}$$

continued



Oscillation Set 2 Pt continued

4

© Use eq (4) and (5) to solve for A.

Divide (5) by ω : $\frac{v_0}{\omega} = -A \sin \phi$ (6)

square (4) and (6) and add.

$$x^2(0) + \left(\frac{v(0)}{\omega}\right)^2 = A^2 \cos^2 \phi + A^2 \sin^2 \phi$$

$$\Rightarrow x^2(0) + \left(\frac{v(0)}{\omega}\right)^2 = A^2$$

$$\Rightarrow A = \left[\frac{d_0^2}{2} + \frac{3}{4} \frac{m_b k d_0^2}{(m_b + m_c)^2} \right]^{1/2}$$

Oscillation – Set 2

1) In figure (a) below, the position versus time of a SHO, estimate the value of the phase angle, ϕ .

$$\frac{\pi}{2} < \phi < \frac{3\pi}{2} \leftarrow \text{in this range}$$

Consider figure (b) below, the velocity versus time of a SHO.

2) Is the particle stationary, moving towards -x, or moving towards +x when the particle is at:

Point A: towards $-x_{max}$

$$x(t) = A \cos(\omega t + \phi)$$

Point B: towards $+x_{max}$

$$v(t) = -A\omega \sin(\omega t + \phi)$$

3) Is $x=0$, $x>0$, or $x<0$ when the particle is at:

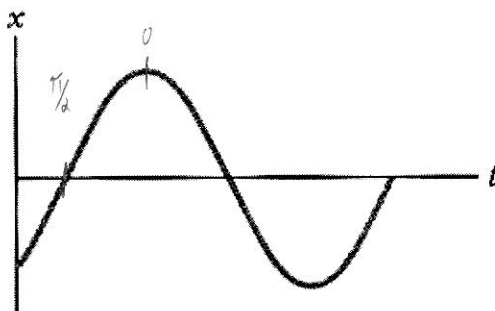
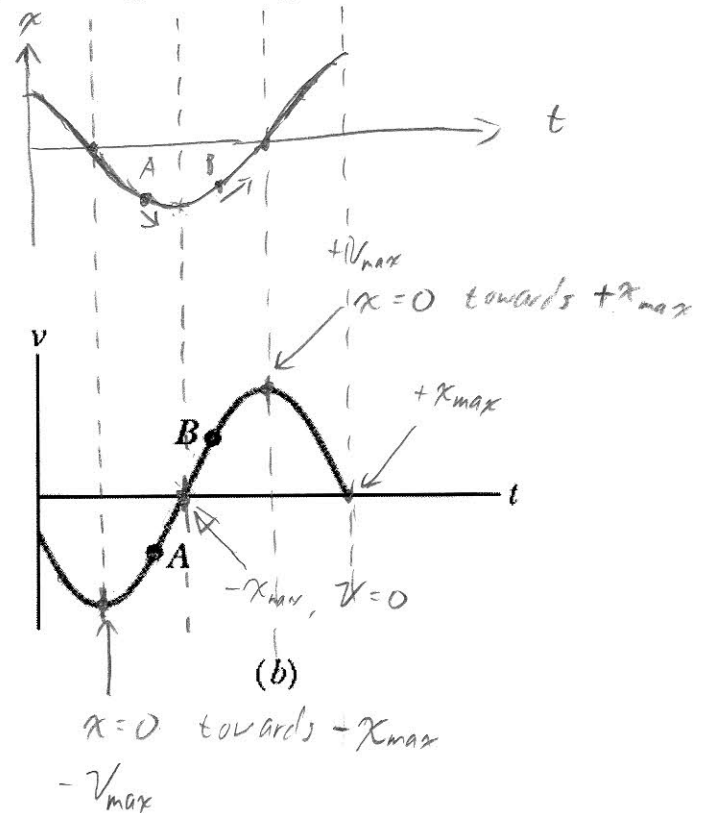
Point A: $x < 0$

Point B: $x < 0$

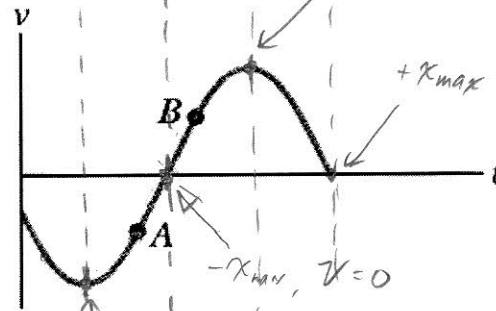
4) Is the particle's speed (the magnitude of it's velocity) increasing, decreasing, or constant when the particle is at:

Point A: Decreasing

Point B: increasing



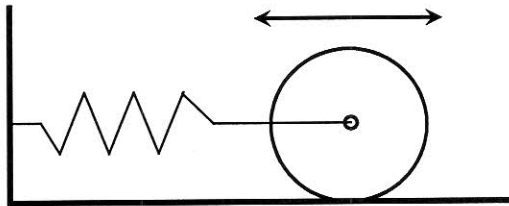
(a)



(b)

Oscillation – Set 2

A solid cylinder of mass M is attached to a horizontal spring with force constant k . The cylinder can roll without slipping along the horizontal plane. When the system is displaced from the equilibrium position, it executes simple harmonic motion. Derive an expression for the period of the oscillations in terms of M, k, I and R .

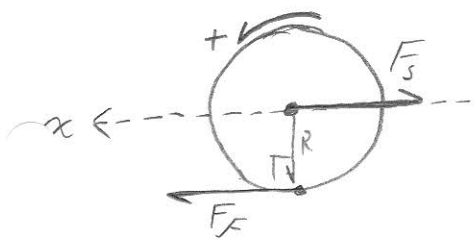


This oscillator is rotating and translating, so we need Torque and translation.

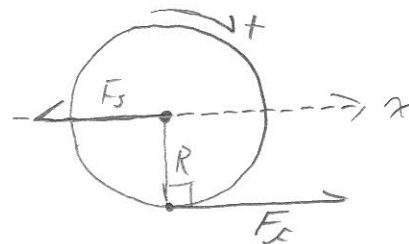
FBD → Pick the initial displacement from equilibrium as positive translation and rotation.

Displace left

Displace right



Just to show that we can do it either way...



$$\textcircled{1} \sum T = I\alpha \Rightarrow \boxed{-F_f R = I\alpha} \leftarrow \text{same} \rightarrow \sum T = I\alpha \Rightarrow \boxed{F_f R = I\alpha}$$

$$\textcircled{2} \sum F = ma \Rightarrow \boxed{F_f - F_s = ma} \leftarrow \text{same} \rightarrow \sum F = ma \Rightarrow \boxed{F_f - F_s = ma}$$

Solve $\textcircled{1}$ for F_f and subst. into $\textcircled{2}$:

$$F_f = -\frac{I}{R}\alpha \xrightarrow{\text{into } \textcircled{2}} -\frac{I}{R}\alpha - kx = ma \Rightarrow -kx = ma + \frac{I}{R^2}a$$

spring ↓
replaced α ↓

$$\Rightarrow \frac{d^2x}{dt^2} = -\left[\frac{k}{\frac{I}{R^2} + m} \right] x$$

ω² ↓

Period ↓
 $T = \frac{2\pi}{\omega} \Rightarrow$

$$\boxed{T = 2\pi \left[\frac{I/R^2 + m}{k} \right]^{1/2}}$$