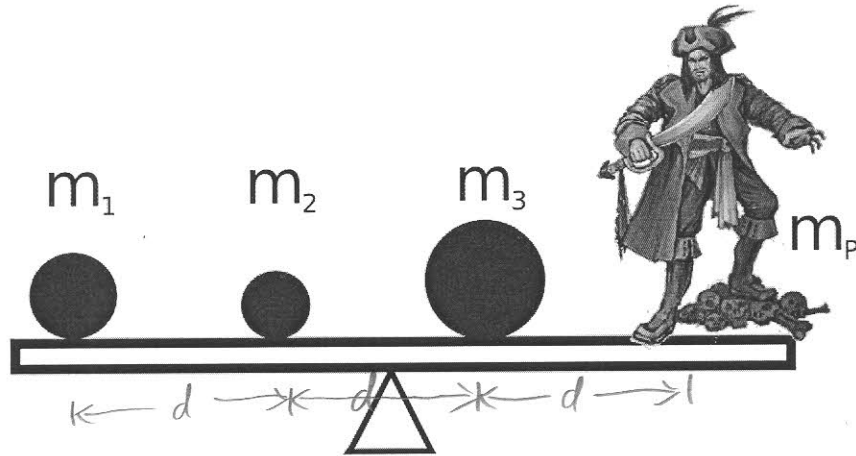


# Systems of Particles – Set 1

1

For reasons that nobody can explain, the Evil Pirate wants to stand on one end of a Plank of Negligible Mass (it's presumably used for walking) and be perfectly balanced by balls from the *Horrible Pendulum of Doom*. He's a little bit OCD, so he insists that all of the objects on the plank must be separated by the same distance  $d$ .



You can avoid walking the Plank of Negligible Mass if you can tell him where to put his fulcrum so that he can stand happily balanced and calm.

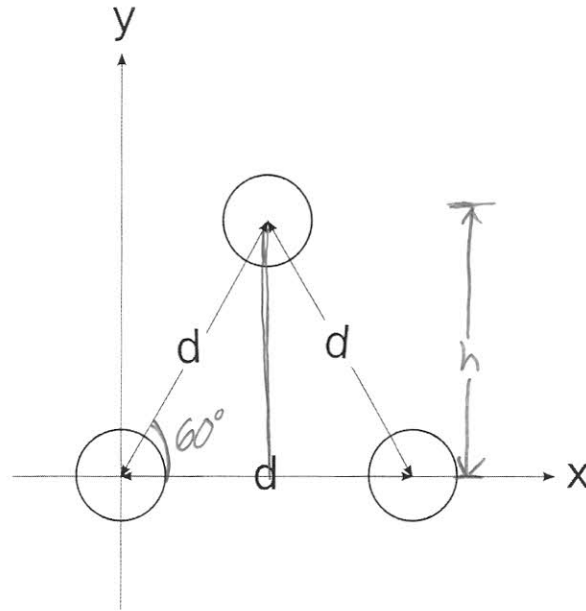
*HINT: Put the fulcrum at the center of mass.*

I'll put  $m_1$  at  $x=0$  and calculate the center of mass using the Mass Weighted Average Position.

$$x_{cm} = \frac{\sum m_i x_i}{\sum m_i} = \frac{m_1(0) + m_2(d) + m_3(2d) + m_p(3d)}{m_1 + m_2 + m_3 + m_p}$$

$$x_{cm} = d \frac{m_2 + 2m_3 + 3m_p}{m_1 + m_2 + m_3 + m_p}$$

Three balls of equal mass form an equilateral triangle. Find the coordinates of the center of mass of the system.



2D problem, calculate  $x_{cm}$  and  $y_{cm}$  separately.

$$x_{cm} = \frac{\sum m_i x_i}{\sum m_i} = \frac{0 \cdot m + \frac{d}{2}m + dm}{3m} = \frac{dm}{3m} \left( \frac{1}{2} + 1 \right) = \frac{d}{3} \cdot \frac{3}{2} = \boxed{\frac{d}{2}}$$

$$y_{cm} = \frac{\sum m_i y_i}{\sum m_i} = \frac{0 \cdot m + 0 \cdot m + hm}{3m}$$

$h = d \sin 60$   
 $\Rightarrow h = d \frac{\sqrt{3}}{2}$

$$y_{cm} = \frac{d \frac{\sqrt{3}}{2} m}{3m} = d \frac{\sqrt{3}}{2 \cdot 3} = \boxed{\frac{d}{2\sqrt{3}}}$$

# Systems of Particles – Set 1

3

We are going to calculate the location of the center of mass of a thin uniform rod of mass  $M$  and length  $L$ . The integral form of the center of mass equation is:

$$l_{cm} = \frac{\int_{l_0}^L l dm}{\int_{l_0}^L dm}$$

The variable of integration is  $l$ , but the differential is  $dm$ . We need to rewrite  $dm$  in terms of  $l$ . In other words, we need to perform a *Change of Variables*. The following steps will walk us through it.

- a. You're in the hardware store and you notice that chain is on sale for \$1.00 per pound. The chain you want is 3 pounds per foot and you want 10 feet. How much do you have to spend?

One to one dollar to pound correspondance, so lb and \$ are the same

$$\left(3 \frac{\text{lb}}{\text{ft}}\right) \cdot 10 \text{ft} = \underline{30 \text{ lb}} \quad \text{or} \quad \underline{\$30}$$

- b. You see that your neighbor is walking out the door with 50 feet of chain. He says gleefully that he paid only 25 dollars for it. What is the *linear density* (mass per unit length) of his chain ?

$$\$25 = 25 \text{ lb}$$

$$\lambda = \frac{25 \text{ lb}}{50 \text{ ft}} = \left[ \frac{1 \text{ lb}}{2 \text{ ft}} \right]$$

linear density

- c. What would you pay for 30 feet of the chain that your neighbor bought?

linear density times length gives mass.

$$\frac{1 \text{ lb}}{2 \text{ ft}} \cdot 30 \text{ft} = 15 \text{ lb} \quad \text{or} \quad \underline{\$15}$$

- d. We generally assign linear density the variable  $\lambda$ . What is the linear density ( $\lambda$ ) of a uniform rod of mass  $M$  and length  $L$ ? (refer to step b for assistance)

Linear density is mass over distance

$$\text{So: } \left[ \lambda = \frac{M}{L} \right]$$

d. Now, write an equation for the mass  $m$  of a piece of this rod in terms of the total mass  $M$ , the total length  $L$ , and the length of the piece  $l$ . (refer to step c for assistance).

linear density times length equals mass ...

$$\underline{m = \lambda l} \Rightarrow \boxed{m = \frac{M}{L} l}$$

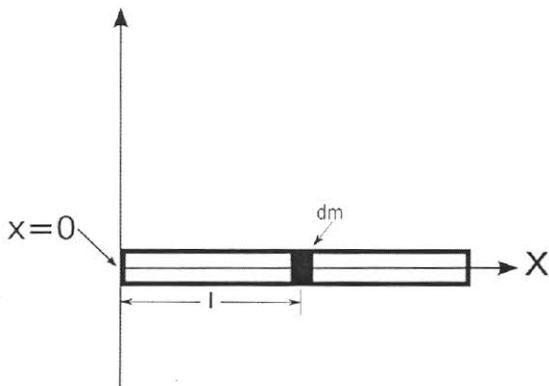
e. Calculate the derivative  $\frac{dm}{dl}$  of the equation above. Now, what is  $dm$  in terms of  $M$ ,  $L$ , and  $dl$ ?

$$\frac{dm}{dl} = \frac{d}{dl} \lambda l = \lambda$$

This is REALLY important

$$\frac{dm}{dl} = \lambda \Rightarrow \boxed{dm = \lambda dl} \Rightarrow \boxed{dm = \frac{M}{L} dl}$$

f. Taking the lower integral first, use the picture below and what we discovered in part e to calculate the total mass. Hopefully you get  $M$ .



$$\int_0^L dm = \int_0^L \frac{M}{L} dl = \frac{M}{L} \int_0^L dl$$

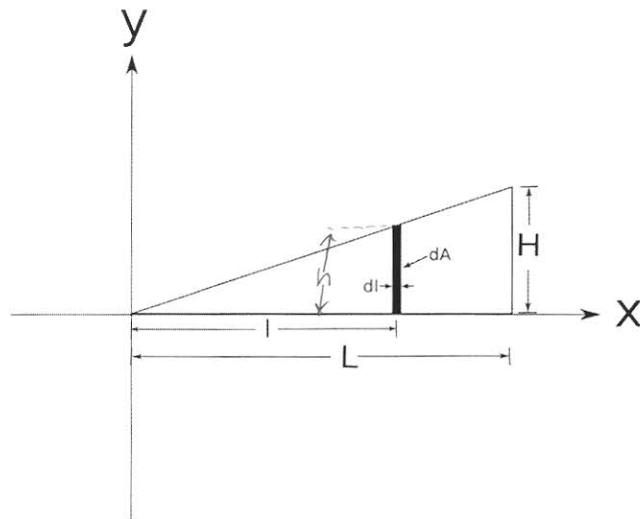
$$= \frac{M}{L} L = \boxed{M}$$

g. Now calculate the the position of the center of mass. Hopefully, you get  $L/2$ .

$$l_{cm} = \frac{\int l dm}{\int dm} = \frac{1}{M} \int_0^L l \frac{M}{L} dl = \frac{1}{L} \int_0^L l dl = \frac{1}{L} \left( \frac{1}{2} l^2 \right) \Big|_0^L = \frac{1}{2} L$$

$$\boxed{l_{cm} = \frac{L}{2}}$$

Calculate the center of mass of a triangular chunk of aluminum of mass  $M$ , length  $l$ , and height  $h$ .



a) Write an expression for  $dm$  in terms of  $dl$  similarly to what we did for the uniform rod.

Now, we consider a slice of the triangle with mass  $dm$  and area  $dA$ .

The slice has height  $h$  and width  $dl$  so:  $dA = h dl$

Linear density  $\lambda$  times length gives mass so...

Surface density ( $\sigma$ ) times area gives mass

$$dm = \sigma dA \Rightarrow dm = \sigma h dl$$

$\sigma$  has units of mass over area so:

$$\sigma = \frac{M}{\frac{1}{2}LH} = \frac{2M}{LH} \Rightarrow dm = \frac{2M}{LH} h dl$$

continued

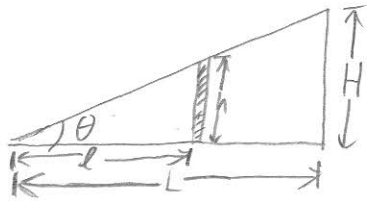


Systems Set 1 - P5 continued

b) calculate  $x_{cm}$

$$x_{cm} = \frac{1}{M} \int_0^L l dm = \frac{1}{M} \int_0^L l \frac{2M}{LH} h dl = \frac{2}{LH} \int_0^L l h dl$$

$h$  depends on  $l$  so we have to rewrite it:



$$\tan \theta = \frac{H}{L}, \quad \tan \theta = \frac{h}{l}$$

$$\text{so: } \frac{H}{L} = \frac{h}{l} \Rightarrow \boxed{h = l \frac{H}{L}}$$

$$x_{cm} = \frac{2}{LH} \int_0^L l \cdot l \frac{H}{L} dl = \frac{2}{L^2} \int_0^L l^2 dl = \frac{2}{L^2} \left( \frac{1}{3} l^3 \right) \Big|_0^L = \frac{2}{L^2} \cdot \frac{1}{3} L^3$$

$$\boxed{x_{cm} = \frac{2}{3} L}$$

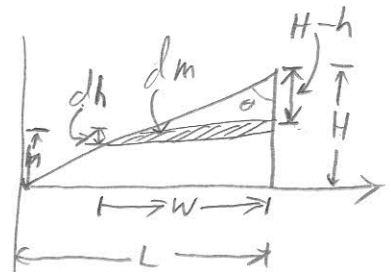
c) calculate  $y_{cm}$

$$y_{cm} = \frac{1}{M} \int_0^H h dm = \frac{1}{M} \int_0^H h \frac{2M}{HL} w dh$$

$$= \frac{2}{HL} \int_0^H h \frac{1}{H} (H-h) dh$$

$$= \frac{2}{H^2} \int_0^H (hH - h^2) dh = \frac{2}{H^2} \left( \frac{1}{2} H^3 - \frac{1}{3} H^3 \right)$$

$$= H \left( 1 - \frac{2}{3} \right) = \boxed{\frac{1}{3} H}$$



$$dm = \sigma dA$$

$$\boxed{dm = \frac{2M}{HL} w dh}$$

$$\text{and } \frac{w}{H-h} = \frac{L}{H}$$

$$\Rightarrow \boxed{w = \frac{L}{H} (H-h)}$$

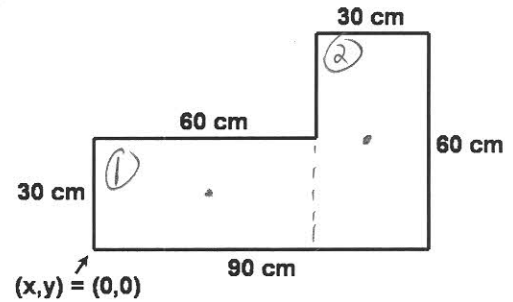
# Systems of Particles – Set 1

The uniform sheet of plywood in the figure has a mass of 20 kg. Find the x and y coordinates of the center of mass.

Given  
 $m = 20$   
 $d_1 = 60$   
 $d_2 = 30$

Want  
 $x_{cm}$   
 $y_{cm}$

$$d_1 = 2d_2$$



HINT: For a perfectly rectangular piece of plywood, the center of mass is exactly in the center of the piece. Imagine that the plywood is composed of two rectangular pieces and then treat those two pieces as point masses located at the center of mass of each piece.

coordinates of center of Each Chunk

Chunk 1

$$x_{cm1} = \frac{d_1}{2}$$

$$y_{cm1} = \frac{d_2}{2}$$

Chunk 2

$$x_{cm2} = d_1 + \frac{1}{2}d_2$$

$$y_{cm2} = \frac{1}{2}d_1$$

$$x_{cm} = \frac{\frac{1}{2}M x_{cm1} + \frac{1}{2}M x_{cm2}}{M} = \frac{1}{2} \left( \frac{d_1}{2} \right) + \frac{1}{2} (d_1 + \frac{1}{2}d_2)$$

$$= \frac{1}{4} 2d_2 + \frac{1}{2} (2d_2 + \frac{1}{2}d_2)$$

$$= \left( \frac{1}{2} + \frac{1}{4} + \frac{1}{2} \right) d_2 = \frac{5}{4} d_2 = \boxed{37.5}$$

$$y_{cm} = \frac{\frac{1}{2}M y_{cm1} + \frac{1}{2}M y_{cm2}}{M} = \frac{1}{2} \frac{d_2}{2} + \frac{1}{2} d_1 = \frac{1}{4} d_2 + \frac{1}{2} 2d_2 = \left( 1 + \frac{1}{4} \right) d_2$$

$$= \frac{5}{4} d_2$$

$$= \boxed{37.5}$$

# Systems of Particles – Set 1

A thin rod of length  $L$  has a linear density of  $\lambda = \lambda_0 \left[ 2 \frac{l^2}{L^2} + \frac{1}{3} \right]$ .

where  $l$  is the distance from one end of the rod and  $\lambda_0$  is a constant with units of mass per unit length.

- a) Calculate the total mass of the rod.  
b) Calculate the center of mass of the rod.

$$\begin{aligned} M &= \int_0^L dm = \int_0^L \lambda dl = \int_0^L \lambda_0 \left[ 2 \frac{l^2}{L^2} + \frac{1}{3} \right] dl \\ &= \lambda_0 \left( \frac{2}{3} \frac{L^3}{L^2} + \frac{1}{3} L \right) = \boxed{\lambda_0 L} \end{aligned}$$

$$\begin{aligned} \text{b) } l_{cm} &= \frac{1}{M} \int_0^L l dm = \frac{1}{\lambda_0 L} \int_0^L l \lambda dl = \frac{1}{\lambda_0 L} \int_0^L l \lambda_0 \left[ 2 \frac{l^2}{L^2} + \frac{1}{3} \right] dl \\ &= \frac{1}{L} \int_0^L \left( \frac{2l^3}{L^2} + \frac{2l}{3} \right) dl = \frac{1}{L} \left( \frac{2}{4} \frac{L^4}{L^2} + \frac{1}{2} \frac{L^2}{3} \right) = L \left( \frac{1}{2} + \frac{1}{6} \right) \end{aligned}$$

$$\boxed{l_{cm} = \frac{2}{3} L}$$

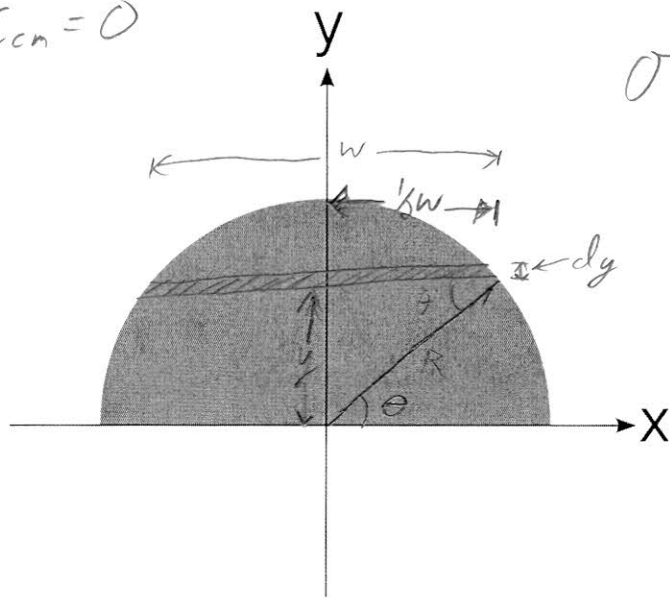


# Systems of Particles – Set 1

6 5

A flat piece of aluminum is cut into a semi-circle of radius  $R$ . Calculate the coordinates of the center of mass.

By symmetry,  $x_{cm} = 0$



$$\sigma = \frac{M}{A} = \frac{M}{\frac{1}{2}\pi R^2}$$

$$y_{cm} = \frac{1}{M} \int y dm, \quad dm = \sigma w dy$$

What's  $w$ ?

Well, according to pythagoras,  $y^2 + (\frac{1}{2}w)^2 = R^2$

$$\Rightarrow \frac{1}{4}w^2 = R^2 - y^2$$

$$\Rightarrow w = 2\sqrt{R^2 - y^2}$$

Okay

$$dm = \frac{2M}{\pi R^2} 2\sqrt{R^2 - y^2} dy$$

continued



Systems of Particles, Set 1, <sup>P5</sup> ~~P6~~ continued

$$y_{cm} = \frac{1}{M} \int_0^R y \frac{4M}{\pi R^2} \sqrt{R^2 - y^2} dy = \frac{4}{\pi R^2} \int_0^R y \sqrt{R^2 - y^2} dy$$

hmmm... how about a substitution?

$$\text{like } u = R^2 - y^2, \text{ then } \frac{du}{dy} = -2y \Rightarrow \underline{dy = -\frac{du}{2y}}$$

$$\text{when: } y = R, u = 0$$

$$y = 0, u = R^2$$

$$\text{So } y_{cm} = -\frac{2M}{\pi R^2} \int_{R^2}^0 y (u)^{\frac{1}{2}} \frac{du}{2y} = \frac{2}{\pi R^2} \int_0^{R^2} u^{\frac{1}{2}} du$$

$$= \frac{2}{\pi R^2} \left( \frac{2}{3} u^{\frac{3}{2}} \right) \Big|_0^{R^2} = \frac{4}{3} \frac{1}{\pi R^2} (R^2)^{\frac{3}{2}} = \frac{4}{3} \frac{R^3}{\pi R^2}$$

$$\boxed{y_{cm} = \frac{4}{3\pi} R}$$