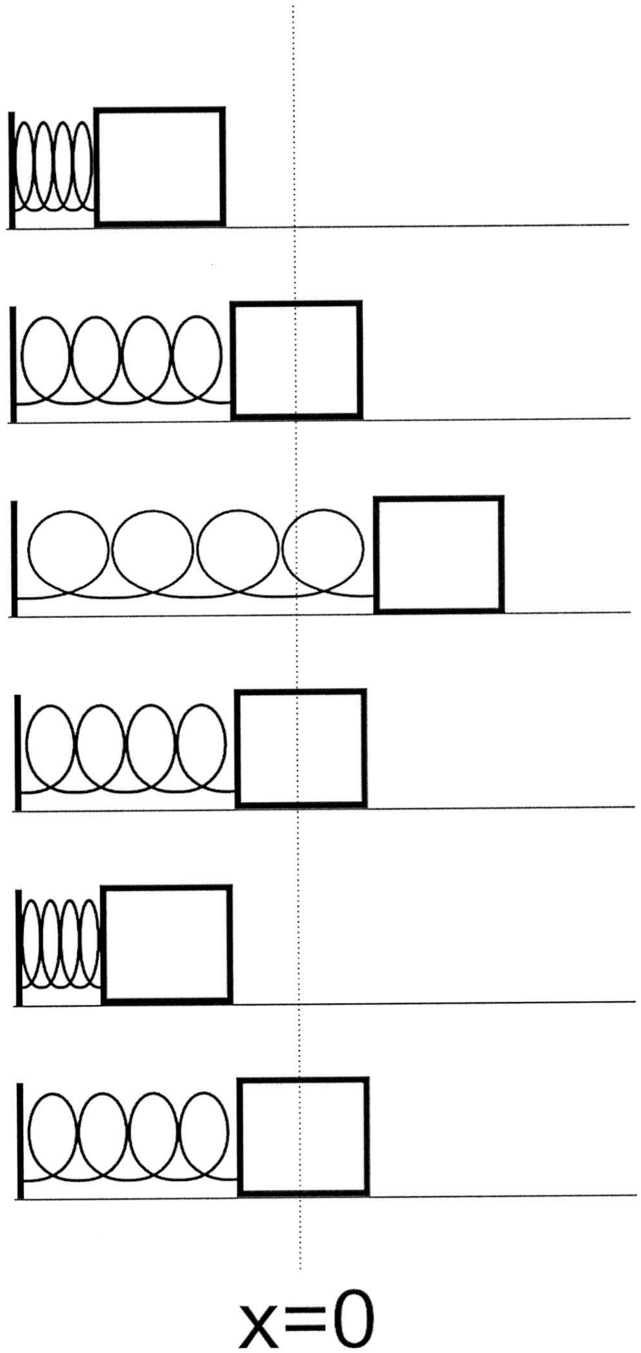
Oscillation – Set 1

Each row in the table below represents a snapshot of a mass attached to a spring. Assume that the mass starts from rest in the first row. In the second row, it is passing through x=0. In the third row, it has reached its maximum extension. In the fifth row, it has reached its maximum compression. In the cells below, mark an arrow indicating the direction of the associated force, acceleration, velocity, and position vectors for each row. If the magnitude is zero, put a zero in the cell.

F	a	v	x
→	→	0	←
0	0	→	0
←	←	0	→
0	0	←	0
→	→	0	←
0	0	→	0



The general form of the differential equation describing a simple harmonic oscillator is:

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

a) Show that the differential equation is satisfied if $x = A\cos(\omega t + \phi)$ where A, ω, and φ are constants.

b) Derive an expression for the velocity of a simple harmonic oscillator.
c) Derive an expression for the acceleration of a simple harmonic oscillator.
HINT: What are the definitions of velocity and acceleration?

d) Why can't we apply the kinematics equations to a simple harmonic oscillator?

a) $$\frac{d^2x}{dt^2} = \frac{d^2}{dt^2}\left(A\cos(\omega t + \phi)\right)$$

$$= \frac{d}{dt}\left[\frac{d}{dt}\left(A\cos(\omega t + \phi)\right)\right]$$

$$= \frac{d}{dt}\left[-\omega A\sin(\omega t + \phi)\right]$$

$$= -\omega^2 \underbrace{A\cos(\omega t + \phi)}_{\Downarrow\ x(t)}$$

$$= \boxed{-\omega^2 x(t)} \qquad \text{QED}$$

b) $$v = \frac{dx}{dt} = \frac{d}{dt}\left(A\cos(\omega t + \phi)\right)$$

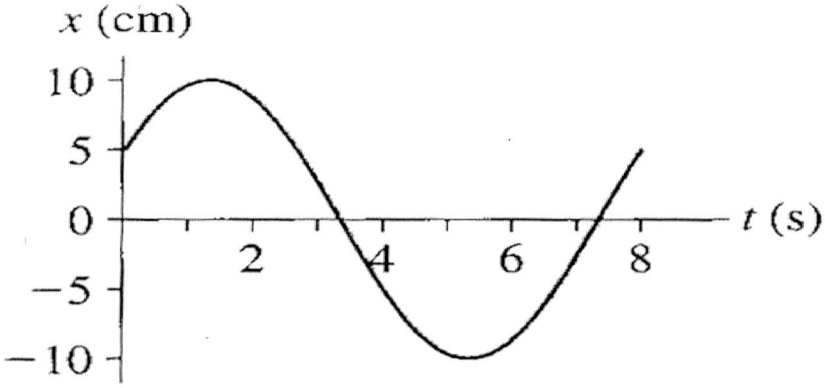
$$\boxed{v = -\omega A\sin(\omega t + \phi)}$$

c) $$a = \frac{dv}{dt} = \frac{d}{dt}\left(-\omega A\sin(\omega t + \phi)\right)$$

$$\boxed{a = -\omega^2 A\cos(\omega t + \phi)}$$

d) a is <u>not</u> constant!

The figure below is a position versus time graph of a particle in simple harmonic motion. Assume that it's position is given by: $x = A\cos(\omega t + \phi)$



a) What is the maximum displacement (amplitude) of the particle?
What is the maximum value of the cos function?
Which constant in the above equation gives the maximum displacement?

max x is 10 cm

max $\cos(\theta)$ is 1

is the max cos is 1, and we multiply by A ...
A gives maximum displacement.

b) What is x when t = 0?
Given your answer to part a, what is the value of φ (known as the phase constant) when t = 0?

Looking at the plot, $x(0) = 5\text{ cm}$

$x(0) = A\cos(\omega(0) + \phi) \Rightarrow 5 = 10\cos(\phi) \Rightarrow \phi = \cos^{-1}(1/2)$
$\phi = 60^\circ$ or $\phi = 1.05$ radians.

c) What is the period (T) of the oscillations?
What are the units of ω? (*HINT: What are the units of the phase angle?*)
What is the mathematical relationship between ω and T?
Find the numerical value of ω.

$T = 8\text{ s}$, ω has units of rad/s, ωt has units of radians

$\omega T = 2\pi$ radians $\Rightarrow \boxed{\omega = \frac{2\pi}{T}} \Rightarrow \omega = \frac{2\pi}{8} = \pi/4$

d) What is the maximum velocity?

$v = -\omega A \sin(\omega t + \phi) = -\omega A = \frac{\pi}{4} 10 = \boxed{\frac{5}{2}\pi}$

e) What is the maximum acceleration?

$a = -\omega^2 A\cos(\omega t + \phi) = \frac{\pi^2}{16} 10 = \boxed{\frac{5}{8}\pi^2}$

Oscillation – Set 1

A block with a mass of m = 2.00 kg is attached to a spring with a spring constant k = 100 N/m. When t = 1.00 s, the position and velocity of the block are x(1s) = 0.129 m and v(1s) = 3.415 m/s.

a) Find the angular frequency, ω, of the oscillator.

b) Find the phase constant.

c) Find the amplitude

d) What was the position of the block at t = 0.00 s?

Step ① – Use NSL to find the oscillator frequency.

FBD

NSL

$$\sum F = ma$$

$$F_s = m\frac{d^2x}{dt^2}\ , \quad F_s = -kx$$

$$\Rightarrow -kx = m\frac{d^2x}{dt^2}$$

$$\Rightarrow \boxed{\frac{d^2x}{dt^2} = -\frac{k}{m}x} \Rightarrow$$ Simple Harmonic Oscillator!

Now, to get the oscillator frequency, we can plug our general solution into the above equation.

General solution: $x(t) = A\cos(\omega t + \phi)$

$$\Rightarrow \frac{d^2}{dt^2}\left[A\cos(\omega t + \phi)\right] = -\frac{k}{m}A\cos(\omega t + \phi)$$

$$\Rightarrow -\omega^2 \cancel{A\cos(\omega t + \phi)} = -\frac{k}{m}\cancel{A\cos(\omega t + \phi)}$$

$$\Rightarrow -\omega^2 = -\frac{k}{m} \Rightarrow \boxed{\omega = \sqrt{\frac{k}{m}}} \Rightarrow \omega = \sqrt{50} = \underline{7.07}$$

Oscillation Set 1, P4 continued

Step 2 – Use Physics to find the initial conditions

In this case, they're given in the problem statement.

at $t = 1.0s$, $\boxed{x(1) = 0.129 m}$, $\boxed{v(1) = 3.415 m/s}$

Step 3 – Use the SHO general solution to solve the rest of the problem.

General Solution:

$$x(t) = A\cos(\omega t + \phi)$$
$$v(t) = -\omega A \sin(\omega t + \phi)$$

Apply our initial conditions

$$x(1) = A\cos(\omega(1) + \phi)$$
$$v(1) = -\omega A \sin(\omega(1) + \phi)$$

Now we can divide to eliminate A and solve for ϕ

$$\frac{v(1)}{x(1)} = \frac{-\omega A \sin(\omega + \phi)}{A\cos(\omega + \phi)} \Rightarrow \frac{v(1)}{x(1)} = -\omega \tan(\omega + \phi)$$

$$\Rightarrow \tan(\omega + \phi) = -\frac{v(1)}{\omega x(1)}$$

$$\Rightarrow \omega + \phi = \tan^{-1}\left(\frac{-v(1)}{\omega x(1)}\right)$$

$$\Rightarrow \boxed{\phi = \tan^{-1}\left(\frac{-v(1)}{\omega x(1)}\right) - \omega}$$

$$\Rightarrow \phi = \tan^{-1}\left(\frac{-3.415}{(7.07)(0.129)}\right) - 7.07 = \boxed{-8.38 \text{ radians}}$$

Oscillations Set 1, P4 continued

Plug our answer for ϕ back into the $x(1)$ eq. and solve for A:

$$x(1) = A\cos(\omega + \phi)$$

$$\Rightarrow \boxed{A = \frac{x(1)}{\cos(\omega + \phi)}}$$

$$\Rightarrow \boxed{A = \frac{0.129}{\cos(7.07 - 8.38)} = 0.5\,m}$$

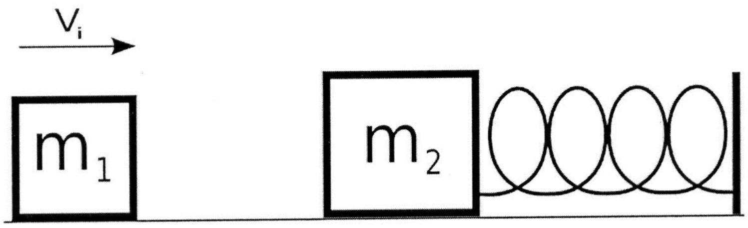
Finally, Put it all together to find $x(0)$

$$x(0) = 0.5\cos(\omega \cdot (0) - 8.38)$$

$$\boxed{x(0) = -0.25\,m}$$

Oscillation – Set 1

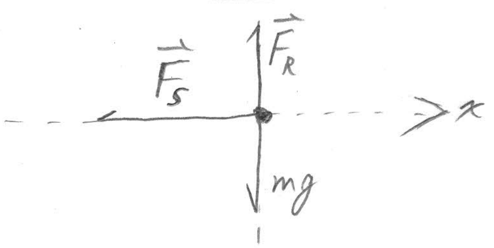
A block with a mass of m_1 = 10 kg is moving to the right with a velocity V_i. It collides and sticks to a block with a mass of m_2 = 15 kg. The second mass is attached to a spring with spring constant k=3 N/m. Before the collision, the spring is at rest in it's equilibrium position.



a) What is the frequency, ~~k~~ ω, of the resulting oscillator after the collision?
b) Assuming that the moment of collision is t=0, find the phase constant of the oscillator?
c) If the resulting amplitude of the oscillator is A = 3 m, what was the initial velocity of m_1?

Step (1) - Use NSL to find ω, the oscillator frequency
The oscillator starts after the collision so:

FBD



NSL

$$\sum \vec{F} = m\frac{d^2x}{dt^2}$$

$$-kx = m_T\frac{d^2x}{dt^2} \Rightarrow \boxed{\frac{d^2x}{dt^2} = -\frac{k}{m}x} \quad (1)$$

General solution is: $\boxed{x(t) = A\cos(\omega t + \phi)}$ (2)

so, combine (1) and (2)

$$\frac{d^2}{dt^2}\left[A\cos(\omega t + \phi)\right] = -\frac{k}{m_T}\left(A\cos(\omega t + \phi)\right)$$

$$\Rightarrow -\omega^2 \cancel{A\cos(\omega t + \phi)} = -\frac{k}{m_T}\left(\cancel{A\cos(\omega t + \phi)}\right)$$

$$\Rightarrow \boxed{\omega = \sqrt{\frac{k}{m}}} \Rightarrow \omega = \left(\frac{3}{(10+15)}\right)^{1/2} \Rightarrow \boxed{\omega = 0.35\ \text{rad}/\text{s}}$$

Oscillation Problems Set 1, P5 continued

Step ② – Use Physics to find the initial conditions

Let: $t=0$ be the moment of collision.

$\boxed{x(t=0)=0}$, starting position is at equilibrium.

Let's find $V(t=0)=V_F$. It's an inelastic collision

$$P_I = P_F$$
$$m_1 V_{1I} = (m_1 + m_2) V_F$$
$$\Rightarrow \boxed{V_F = \frac{m_1}{m_1+m_2} V_{1I}}$$

Step ③ – Use the general solution of a SHO to solve the problem.

Part b asks for the phase constant.

Let's find it.

In general:

$$x(t) = A\cos(\omega t + \phi)$$
$$v(t) = -\omega A \sin(\omega t + \phi)$$

$$\Rightarrow 0 = A\cos(\phi) \Rightarrow \cos(\phi) = 0$$

True when $\phi = \pi/2,\ 3/2\,\pi$

continued

↓

Oscillation Problems Set 1, P5 continued

But we need $v(0)$ to be positive:

$$v(0) = -\omega A \sin(\phi), \quad \sin\left(\frac{\pi}{2}\right) = 1, \quad \sin\left(\frac{3\pi}{2}\right) = -1$$

so, $\boxed{\phi = 3\pi/2}$

Now, we'll use the velocity equation to find v_{1I}

$$v(0) = v_F = -\omega A \sin(\phi), \quad \phi = 3\pi/2 \text{ so } \sin(\phi) = -1$$

$$\Rightarrow \frac{m_1}{m_1 + m_2} v_{1I} = \omega A$$

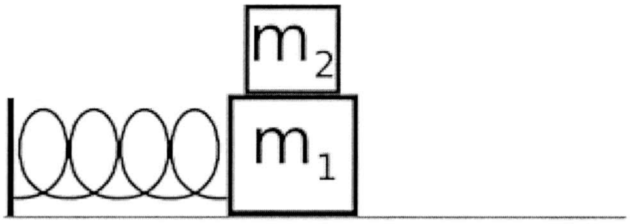
$$\Rightarrow \boxed{v_{1I} = \frac{m_1 + m_2}{m_1} \omega A}$$

$$v_{1I} = \frac{10 + 15}{10} (0.35)(3)$$

$$\boxed{v_{1I} = 2.6 \text{ m/s}}$$

Oscillation – Set 1 6

Two blocks, m_1=10 kg and m_2=1 kg, and a spring (k=200 N/m) are arranged on a horizontal, frictionless surface. The coefficient of static friction between the blocks is μ_s=0.40. What is the maximum amplitude of the resulting simple harmonic oscillator if m_2 is not to slip?



Step ① - Use NSL to find the oscillator Freq.

Since m_1 and m_2 are "fused" together, They act as a single mass as far as the spring is concerned.

FBD

NSL

$$\Sigma F = ma$$

$$F_s = (m_1 + m_2)\frac{d^2x}{dt^2}$$

$$-kx = (m_1 + m_2)\frac{d^2x}{dt^2}$$

$$\Rightarrow \frac{d^2x}{dt^2} = -\underbrace{\left[\frac{k}{m_1+m_2}\right]}_{\omega^2} x \Rightarrow \boxed{\omega = \left(\frac{k}{m_1+m_2}\right)^{1/2}}$$

continued

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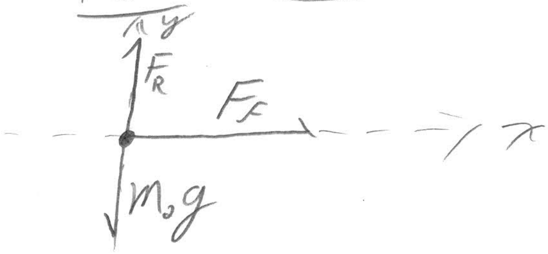
Oscillation Set 1, P6 continued

step② - Use physics to get the initial conditions.

The condition for non-slipping: $\underline{F_f = \mu_s F_R} \leq ma$

Let's use NSL to find a_{max}

FBD - For m_2



NSL

$\sum \vec{F} = m_2 \vec{a}$

$x: F_f = m_2 a_x$

$\boxed{\mu_s F_R = m_2 a_{max}}$ ①

$y: F_R - m_2 g = \cancel{m a_y}^{\,0}$

$\Rightarrow \boxed{F_R = m_2 g}$ ②

Remember, $\mu_s F_R$ is the maximum force that static friction can provide, thus, $\mu_s F_R = m_2 \underline{a_{max}}$, maximum acceleration before slipping.

Combine ① and ②:

$$\mu_s \cancel{m_2} g = \cancel{m_2} a_{max} \Rightarrow \boxed{a_{max} = \mu_s g}$$

So, for initial conditions, let's let $t=0$

Then: $a(0) = a_{max} \Rightarrow \boxed{a(0) = \mu_s g}$

In a simple harmonic oscillator, when $a = \underline{a_{max}}$, $\underline{x = x_{max}}$

So: $\boxed{x(0) = A}$

continued ↓

Oscillation Set 1, P6 continued

Step 3 - Use SHO General solution to solve the problem.

$$x(t) = A\cos(\omega t + \phi)$$
$$v(t) = -\omega A \sin(\omega t + \phi)$$
$$a(t) = -\omega^2 A \cos(\omega t + \phi)$$

In the initial condition, we got $a(0)$ and $x(0)$

$$x(0) = A\cos(\phi)$$
$$a(0) = -\omega^2 A\cos(\phi)$$

Let's divide to eliminate $\cos(\phi)$

$$\frac{x(0)}{a(0)} = \frac{\cancel{A\cos\phi}}{-\omega^2 \cancel{A\cos\phi}} \Rightarrow \frac{A}{\mu_s g} = \frac{-1}{\omega^2}$$

$A = -\frac{\mu_s g}{\omega^2}$, now A is generally positive and the sign can be fixed with the right choice for ϕ so...

$$\boxed{A = \frac{\mu_s g}{\omega^2}} \Rightarrow \boxed{A = \frac{\mu_s g}{k}(m_1 + m_2)}$$

$$A = \frac{(0.4)(9.8)}{200}(10+1) = \boxed{0.22\ m}$$