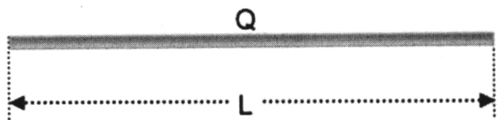
Name: ____________________ **Problems Solved** __/ 9

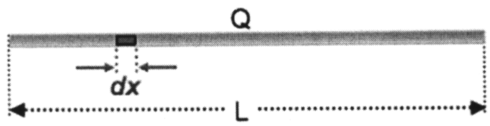
A total charge Q is uniformly distributed over the length L of a line charge distribution. The charge density λ is given by

a) Q/L
b) $(Q/L)dx$
c) L/Q
d) Qdx



A total charge Q is uniformly distributed over the length L of a line charge distribution. The total charge inside a short element dx is given by

a) Q/L
b) $(Q/L)dx$
c) L/Q
d) Qdx



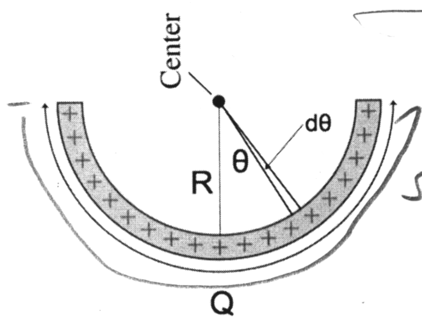
$dq = \lambda dx$

$dq = \frac{Q}{L} dx$

A total charge Q is uniformly distributed over a half ring with radius R. The charge density λ is given by

a) $Q/2\pi R$
b) $Q/\pi R$
c) R/Q
d) $\pi R/Q$

$\lambda = \frac{Q}{S} = \frac{Q}{\pi R}$



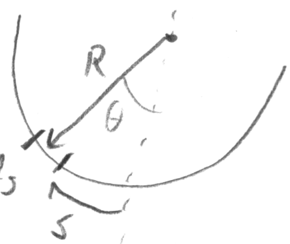
$S = \pi R$

A total charge Q is uniformly distributed over a half ring with radius R, as above. The total charge inside a small element dθ is given by:

a) $(Q/2\pi)d\theta$
b) $(Q/\pi)d\theta$
c) $(Q/2\pi R)dR$
d) $(Q/\pi R)dR$

$q = \lambda ds$

$\Rightarrow \boxed{q = \frac{Q}{\pi R} R d\theta}$ ds



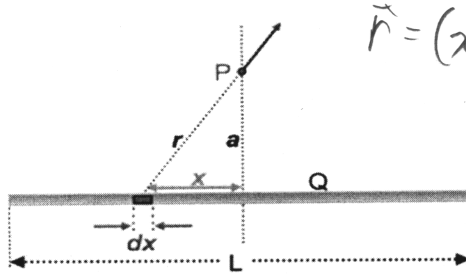
$S = R\theta$

$\Rightarrow \frac{ds}{d\theta} = R \Rightarrow ds = R d\theta$

A total charge Q is uniformly distributed over the length L of a line charge distribution. The $\hat{j}$ (vertical) component of electric field at point P created by a short element dx is given by:

a) $\frac{kQa}{Lr^3}dx$ b) $\frac{kQx}{Lr^3}dx$ c) $\frac{kQa}{Lr^2x}dx$ d) $\frac{kQx}{Lr^2a}dx$

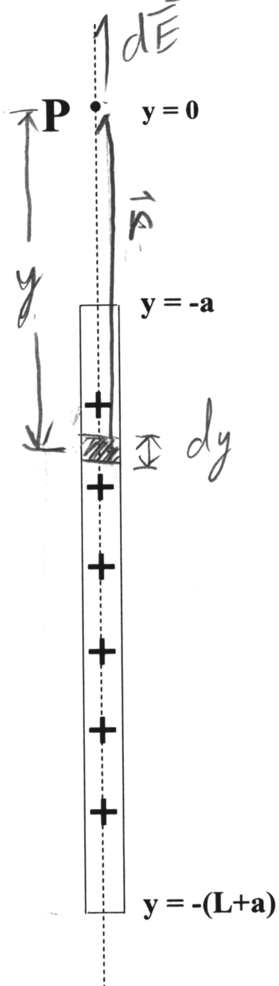
$\vec{r} = (x\hat{i} + a\hat{j})$



$d\vec{E} = \frac{k\,dq}{r^2}\hat{r} = \frac{k\frac{Q}{L}dx}{r^2}\frac{\vec{r}}{r}$

$= \frac{kQa}{Lr^3}$

Find the net electric field vector at point P due to a uniformly charged rod of total charge Q and length L. Point P is at the origin, a distance a from one end of the line.



$$\vec{E} = \int d\vec{E}$$

$$= \int \frac{k\,dq}{|\vec{r}|^2}\hat{r} \qquad dq = \lambda\,dy, \quad \lambda = \frac{Q}{L}$$

$$= \int_{-(L+a)}^{-a} \frac{k\frac{Q}{L}dy}{y^2}\hat{j} = \frac{kQ}{L}\int_{-(L+a)}^{-a} \frac{1}{y^2}\,dy\,\hat{j}$$

$$= \frac{kQ}{L}\left(-\frac{1}{y}\right)\Big|_{-(L+a)}^{-a} = \frac{kQ}{L}\left(\frac{1}{a} - \frac{1}{L+a}\right)\hat{j}$$

$$= \frac{kQ}{L}\left[\frac{L+\cancel{a}-\cancel{a}}{a(L+a)}\right]\hat{j} = \frac{kQ}{\cancel{L}}\,\frac{\cancel{L}}{a(L+a)}\hat{j}$$

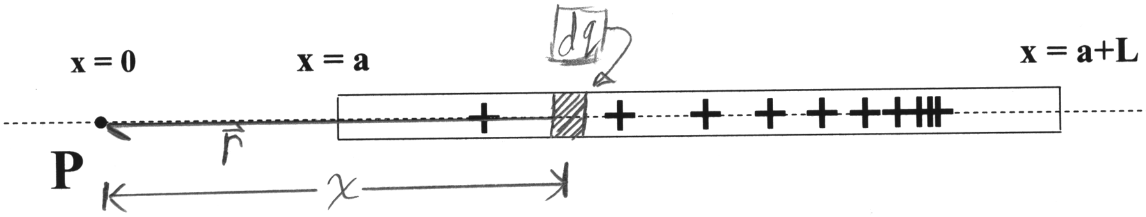
$$\vec{r} = y\hat{j}$$

$$\hat{r} = \hat{j}$$

$$\Rightarrow \boxed{\vec{E} = \frac{kQ}{a(L+a)}\hat{j}}$$

A charged rod of length L has a non-uniform linear charge density $\cancel{\lambda = 3\frac{x^3}{L^3}\ \text{C/m}}$. $\lambda = \lambda_0 \frac{x^3}{L^3}\ \text{C/m}$

a) Calculate the Total Charge, Q, of the rod.
b) Calculate the net electric field vector at the point P shown in the figure.



$$\vec{r} = -x\hat{\imath},\quad \hat{r} = -\hat{\imath},\quad |\vec{r}| = x,\quad dq = \lambda\, dx$$

a) $$Q = \int dq \Rightarrow Q = \int_a^{a+L} \lambda\, dx$$

$$Q = \int_a^{a+L} \lambda_0 \frac{x^3}{L^3} dx = \left(\lambda_0 \frac{3}{4}\frac{x^4}{L^3}\right|_a^{a+L} = \boxed{\frac{3}{4}\frac{\lambda_0}{L^3}\left((a+L)^4 - a^4\right)}$$

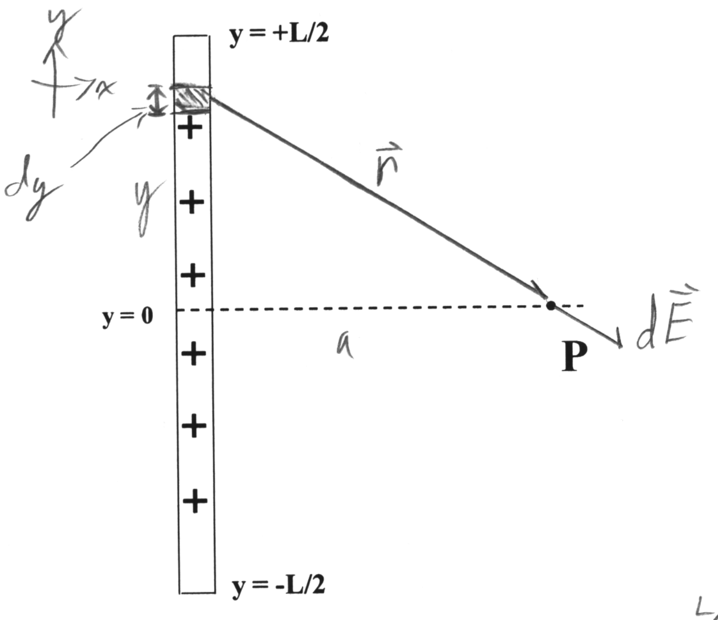
b) $$\vec{E} = \int d\vec{E} = \int \frac{k\,dq}{|\vec{r}|^2}\hat{r} = \int_a^{a+L} \frac{k\left(\lambda_0 \frac{\cancel{x^3}}{L^3} dx\right)}{\cancel{x^3}}(-x\hat{\imath})$$

$$= \frac{-k\lambda_0}{L^3}\int_a^{a+L} x\, dx\, \hat{\imath} = -\frac{k\lambda_0}{L^3}\left(\tfrac{1}{2}x^2\right|_a^{a+L}\hat{\imath} = -\frac{k\lambda_0}{L^3}\left((a+L)^2 - a^2\right)\hat{\imath}$$

$$= -\frac{k\lambda_0}{L^3}\left(\cancel{a^2} + 2aL + L^2 - \cancel{a^2}\right)\hat{\imath}$$

$$\vec{E} = -\frac{k\lambda_0 \cancel{L}}{L^{\cancel{3}\,2}}(2a + L)\hat{\imath} \Rightarrow \boxed{\vec{E} = \frac{k\lambda_0}{L^2}(2a+L)\hat{\imath}}$$

Find the net electric field vector at point P due to a uniformly charged rod of total charge Q and length L. Point P is a distance a from the rod, and it lies along a line that perpendicularly bisects the rod.



$$\vec{r} = a\hat{\imath} + y\hat{\jmath}$$

$$|\vec{r}| = (a^2 + y^2)^{1/2}$$

$$\lambda = \frac{Q}{L}, \quad dq = \frac{Q}{L}\,dy$$

$$\vec{E} = \int d\vec{E} = \int \frac{k\,dq}{|\vec{r}|^2}\hat{r} = \int_{-L/2}^{L/2} \frac{k\frac{Q}{L}dy}{(a^2+y^2)^{3/2}}\,(a\hat{\imath} + y\hat{\jmath})$$

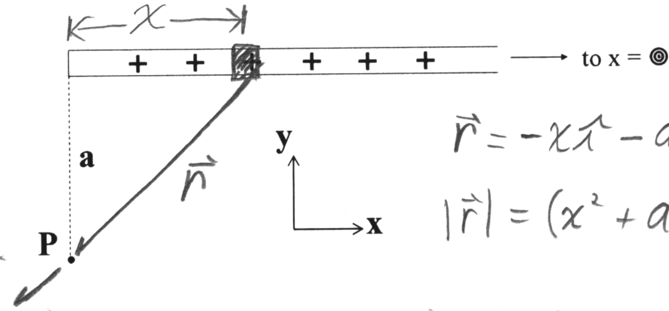
* By symmetry, the $\hat{\jmath}$ component sums to zero

magic integral machine

$$\Rightarrow \vec{E} = \frac{kQa}{L}\int_{-L/2}^{L/2} \frac{\hat{\imath}}{(a^2+y^2)^{3/2}}\,dy \Rightarrow \text{Poof!} \Rightarrow \frac{kQ}{aL}\left(\frac{y}{(y^2+a^2)^{1/2}}\right)\Bigg|_{-L/2}^{L/2}\hat{\imath}$$

$$\vec{E} = \frac{kQ}{a\cancel{L}}\left[\frac{2\,\cancel{L}/2}{\left(\frac{L^2}{4}+a^2\right)^{1/2}}\right]\hat{\imath} \Rightarrow \boxed{\vec{E} = \frac{kQ}{a(a^2+L^2/4)^{1/2}}\,\hat{\imath}}$$

1. A **semi-infinite** (i.e., infinite in one direction) plastic charged rod has uniform charge density λ. Starting from scratch, find the net electric field vector at point P, a distance a beneath one end of the rod. Show all work. Write your answer in terms of λ, a, and other constants.



$$\vec{r} = -x\hat{\imath} - a\hat{\jmath}$$
$$|\vec{r}| = (x^2 + a^2)^{1/2}$$

~~Hint: E_{tot} has both x and y components here because there's no symmetry. Calculate E_x and E_y separately and then combine the vectors.~~ Also,

$$\int \frac{x\,dx}{\left(x^2+a^2\right)^{3/2}} = \frac{-1}{\sqrt{x^2+a^2}}$$

$$\int \frac{dx}{\left(x^2+a^2\right)^{3/2}} = \frac{1}{a^2}\frac{x}{\sqrt{x^2+a^2}}$$

$$\lim_{x\to\pm\infty} \frac{x}{\sqrt{x^2+a^2}} = \pm 1 \text{ (think of a} \to 0\text{)}$$

$$\vec{E} = \int d\vec{E} = \int \frac{k\,dq}{|\vec{r}|^2}\hat{r}\ , \quad dq = \lambda\,dx$$

$$\Rightarrow \vec{E} = \int_0^\infty \frac{k\lambda\,dx}{(x^2+a^2)^{3/2}}(-x\hat{\imath} - a\hat{\jmath})$$

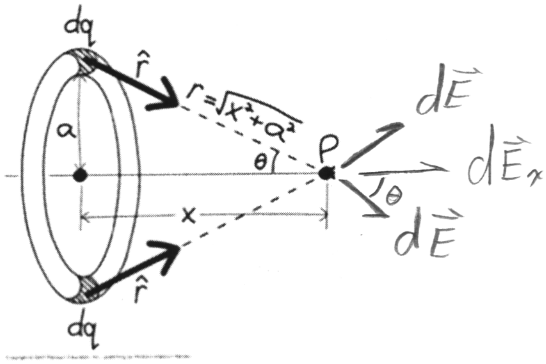
$$= -k\lambda\left[\int_0^\infty \frac{x}{(x^2+a^2)^{3/2}}dx\,\hat{\imath} + \int_0^\infty \frac{a}{(x^2+a^2)^{3/2}}dx\,\hat{\jmath}\right]$$

$$= -k\lambda\left[\left(\frac{-1}{(x^2+a^2)^{1/2}}\right|_0^\infty \hat{\imath} + \left(\frac{xa}{a^2(x^2+a^2)^{1/2}}\right|_0^\infty \hat{\jmath}\right]$$

$$= -k\lambda\left[\left(0 + \frac{1}{a}\right)\hat{\imath} + \left(\frac{1}{a} - 0\right)\hat{\jmath}\right]$$

$$\Rightarrow \boxed{\vec{E} = -\frac{k\lambda}{a}\hat{\imath} - \frac{k\lambda}{a}\hat{\jmath}}$$

A ring of radius *a* carries a charge *Q* distributed evenly over the ring. Find an expression for the electric field at any point on the axis of the ring.



All components other than the $\hat{x}$ component cancel.

So, we'll only consider the x-axis

$$d\vec{E}_x = |d\vec{E}|\cos\theta\,\hat{x}, \quad |d\vec{E}| = \frac{k\,dq}{|\vec{r}|^2}, \quad \cos\theta = \frac{x}{|\vec{r}|}$$

$$\vec{E} = \int d\vec{E}_x = \int \frac{k\,dq}{|\vec{r}|^2}\frac{x}{|\vec{r}|}\hat{x}$$

$$= \int \frac{kx}{(a^2+x^2)^{3/2}}\,dq\,\hat{x}$$

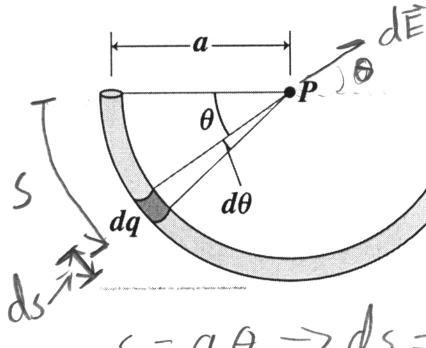
$$= \frac{kx}{(a^2+x^2)^{3/2}} \int dq\,\hat{x}$$

This is constant

This is Just Q!

$$\Rightarrow \boxed{\vec{E} = \frac{kQx}{(a^2+x^2)^{3/2}}\hat{x}}$$

A semicircular loop of radius a carries a positive charge Q distributed uniformly over its length. Find the electric field at the center of the loop (point P in the figure). *Hint:* Divide the loop into charge elements dq as shown, and write dq in terms of the angle $d\theta$. Then integrate over θ.



In this instance, Polar coordinates work nicely, as $|\vec{r}|$ is constant.

$\hat{r} = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$

$|\vec{r}| = a$

$s = a\theta \Rightarrow ds = a\,d\theta$

$$\vec{E} = \int d\vec{E}$$

$$= \int \frac{k\,dq}{|\vec{r}|^2}\hat{r}, \quad dq = \lambda\,ds, \quad \lambda = \frac{Q}{\pi a}$$

$$\Rightarrow dq = \frac{Q}{\pi \cancel{a}}\cancel{a}\,d\theta \Rightarrow \boxed{dq = \frac{Q}{\pi}d\theta}$$

$$\Rightarrow \vec{E} = \frac{k}{a^2}\int_0^{\pi} \frac{Q}{\pi}(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})\,d\theta$$

$$= \frac{kQ}{\pi a^2}\left[\underbrace{\int_0^{\pi}\cos\theta\,d\theta}_{0}\,\hat{\imath} + \underbrace{\int_0^{\pi}\sin\theta\,d\theta}_{2}\,\hat{\jmath}\right]$$

$$\boxed{\vec{E} = \frac{2kQ}{\pi a^2}\hat{\jmath}}$$